Ch 13 Vector-Valued Functions And Motion In Space

The functions we worked with, so far, are called real-valued functions ($y=f(x)$). In them, the domain (input) "x" is a real number as will as the range (output) "y".

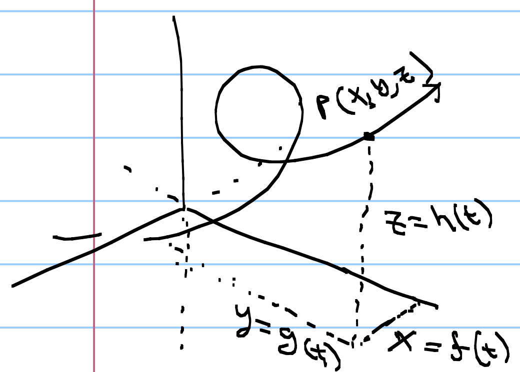
In this chapter we will study vector-valued functions.

$$r(t) = \langle f(t), g(t), h(t) \rangle \quad \text{(in space)}$$

In them, the domain "t" is a real number, but the range (output) (value of the function) "r" is a vector.

We will use vector-valued funs to descripe the paths and motions of objects in space or plane and study their proerties (velcity, Acceleratio, turn and twist)

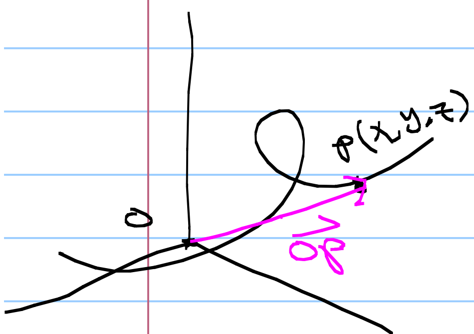
13.1 Curves in Space and Their Tangents



A curve in space can be thought of as the path of a particle whose coordinates (x,y,z) are function of time "t" $x=f(t)$, $y=g(t)$, $z=h(t)$

$t \in I_{\text{interval}}$

These eqns parametrize the curve (they represent the curve)

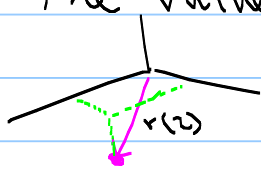


Another representation of the curve is the vector form

$$\vec{OP} = r(t) = \langle f(t), g(t), h(t) \rangle$$
$$= f(t)\,i + g(t)\,j + h(t)\,K$$

Note $r(t)$ is a vector-valued fun
$f(t), g(t), h(t)$ are real-valued funs
(scalar functions)

Ex the value of $r(t) = 3t\,i + (t^2 - t)\,j - 3K$ at $t = 2$ is



$r(2) = 6i + 2j - 3K = \langle 6, 2, -3 \rangle$

at different t's, the vector points to different points making up the curve.

Examples of curves in space. use maple to graph

$r(t) = (\sin 3t \cos t)\,i + (\sin 3t \sin t)\,j + t\,K$

$r(t) = (\cos t)\,i + (\sin t)\,j + (\sin 2t)\,K$

$r(t) = (4 + \sin 20t)(\cos t)\,i + (4 + \sin 2t)(\sin t)\,j + (\cos 20t)\,K$

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with(VectorCalculus)
SpaceCurve(<f(t),g(t),h(t)>,t=a..b)
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(Note: this is don by evaluating many points and connecting them)

Without softwares, we need previous knowllege

Ex describe the curve defined by the vector function
$r(t) = \langle 1+t, 2+5t, -1+6t \rangle$. The corresponding parametric eqns

$x = 1+t \quad y = 2+5t \quad z = -1+6t$, from 12.5, are for the line through $P(1, 2, -1)$ parallel to $v = \langle 1, 5, 6 \rangle$

Ex 1 page 708 Graph the Vector fun

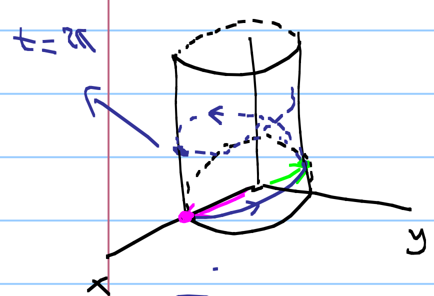
$$r(t) = (\cos t)i + (\sin t)j + tK$$

write the parametric eqns and chose two for a recognized surface, the curve will be on the surface. Vary the third eqn to follow the curve.

$x = \cos t \quad y = \sin t \quad z = t$ we know $\cos^2 + \sin^2 = 1$

$\Rightarrow x^2 + y^2 = 1$ in space

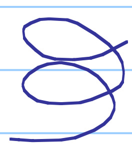
this is a cylinder



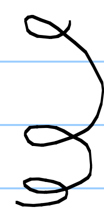
$t = 0 \quad r(0) = \langle 1, 0, 0 \rangle$

$t = \frac{\pi}{2} \quad r(\frac{\pi}{2}) = \langle 0, 1, \frac{\pi}{2} \rangle$

$t = \pi \quad r(\pi) = \langle -1, 0, \pi \rangle$



helix (spiral)



? what if $z = t^2$? the curve gose up not linearly (Not a helix)

? How about $r(t) = \langle \sin t, \cos t, t \rangle$ spiral clockwize

? How about $r(t) = ti + \sin t\, j + \cos t\, K$ helix along x-axis

Limits and Continuity

Limits of vector-valued funs are defined similarly as real-valued funs.

From the definition, if $r(t) = f(t)i + g(t)j + h(t)K$

then $\lim_{t\to t_0} r(t) = \left(\lim_{t\to t_0} f(t)\right)i + \left(\lim_{t\to t_0} g(t)\right)j + \left(\lim_{t\to t_0} h(t)\right)K$

if all the components limits exist

Ex 2 page 709

If $r(t) = \cos t\, i + \sin t\, j + t\, K$

$\lim_{t\to \frac{\pi}{4}} r(t) = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j + \frac{\pi}{4} K$ We take limit component by component

Continuity (similar for real-valued fun)

$r(t)$ is continuous at $t = t_0$ if

1) $r(t_0)$ defined 2) $\lim_{t\to t_0} r(t)$ exist 3) $\underbrace{r(t_0) = \lim_{t\to t_0} r(t)}_{\text{include 1) and 2)}}$

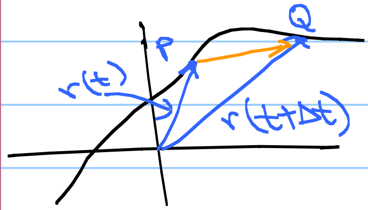
Note: from def of $\lim_{t\to t_0} r(t)$, $r(t)$ is cont <u>iff</u> each component scalar fun is continuous.

Ex 3 page 709

a) continuous because the component's fun are go over it

b) $r(t) = \cos t\, i + \sin t\, j + \lfloor t \rfloor K$ is continuous for $t \neq$ integer

Derivative and Motion



Suppose the curve in space $r(t) = f(t)i + g(t)j + h(t)K$ represents the path of a particle then the difference between the particles position at time t and time $t+\Delta t$ is

$$\Delta r = r(t+\Delta t) - r(t)$$
$$= f(t+\Delta t)i + g(t+\Delta t)j + h(t+\Delta t)K - f(t)i - g(t)j - h(t)K$$
$$= (f(t+\Delta t) - f(t))\,i + (g(t+\Delta t) - g(t))\,j + h(t+\Delta t)$$

As $\Delta t \to 0$
1) Q approach P along the curve
2) The secant line PQ becomes tangent to the curve at P
3) $\frac{\Delta r}{\Delta t}$ Approaches the limit

$$\lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}\, i + \lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t}\, j + \lim_{\Delta t \to 0} \frac{h(t+\Delta t) - h(t)}{\Delta t}\, K$$

$$= f'(t)i + g'(t)j + h'(t)K$$ this is the def of the derivative of $r = r(t)$

Definition:

If $r(t) = f(t)i + g(t)j + h(t)K$ then the derivative of $r(t)$ is $\frac{dr}{dt} = r'(t) = f'(t)i + g'(t)j + h'(t)K$

provided $f'(t)$, $g'(t)$, $h'(t)$ defined

- If r' is continuous and never $\mathbf{0} = \langle 0,0,0 \rangle$ the r is smooth
- $r'(t)$ at p is the vector tangent to the curve at p
- the tangent line to the curve at p is the line through p in the direction of the vector tangent.
- a curve is piecewise smooth if it is made up of finite smooth curves pieced together in a continuous fashion

Exercise 19 page 714

If $r(t) = \sin t\, i + (t^2 - \cos t)\, j + e^t K$ then find

1) $r'(t)$ 2) the tangent vector to the curve at $t_0 = 0$

3) the tangent line to the curve at $t_0 = 0$

1) $r'(t) = \cos t\, i + (2t + \sin t)\, j + e^t K$

2) $r'(0) = i + K$

3) point is $(\sin(0), (0)^2 - \cos(0), e^0) = (0, -1, 1)$

direction vector is $\langle 1, 0, 1 \rangle$

$\therefore$ eqns $x = 0 + 1t$ $\quad y = -1 + 0t$ $\quad z = 1 + 1t$

eqns of line through $p(x_0, y_0, z_0)$ in the direction $v = \langle v_1, v_2, v_3 \rangle$ are
$x = x_0 + v_1 t$
$y = y_0 + v_2 t$
$z = z_0 + v_3 t$

Graph the curve and the line $r(t) = \langle t, -1, 1+t \rangle$ then copy and paste one on the other in maple.
See if you can graph vectors

Derirative and Motion

Defs:

If $r(t)$ is the posision vector of a particle moving a long a smooth curve in space (includes plane) then

$V(t) = r'(t)$ is the particle's velocity { Quantity with Magnitude & direction

$|V| =$ speed $\quad \frac{V}{|V|} =$ direction of motion

$a = V' = r''(t)$ is the acceleration

Ex 4 page 711 $\quad r(t) = 2\cos t\, i + \sin t\, j + 5\cos^2 t\, K$

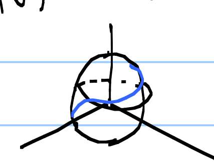
Differentiation Rules page 712. go over them and Note

$\frac{f(x)}{g(x)} = \frac{\text{real}}{\text{real}}$ is defined. for vectors No qoutient Rule

$f(x) * g(x) = \text{real} * \text{real}$ is defined. for vectors we have $V_1 \cdot V_2$ or $V_1 \times V_2$

vector functions of constant length (speed) ($|V| = c$)

if $r(t)$ is on a sphere at the Origion then



$|V| = c \Rightarrow |r(t)| = c \Rightarrow r(t) \cdot r(t) = c^2$

$(r \cdot r = |r|^2)$

differentiate both sides

$\Rightarrow \frac{d}{dt}(r(t) \cdot r(t)) = 0 \Rightarrow r'(t) \cdot r(t) + r(t) \cdot r'(t) = 0$

from Rules

$\Rightarrow 2r'(t) \cdot r(t) = 0 \Rightarrow r'(t) \cdot r(t) = 0 \Rightarrow r'(t) \perp r(t)$

(Vector tangent $\perp$ position vector)

If r is diff of constant length then $r \cdot \frac{dr}{dt} = 0$

we will use this in 13-4

13.2 Integrals of vector functions; $\underbrace{\text{Projectile Motion}}_{\text{Application}}$

If $R'(t) = r(t)$ then R is an antiderivative of r
adding $\vec{C}$ to R and differentiate $\frac{d}{dt}(R+C) = r(t)$

$\therefore$ The indefinit integral of r is $\int r(t)\,dt = R(t) + \vec{C}$ $\quad \vec{C} = \langle c_1, c_2, c_3 \rangle$

$$\int f(t)i + g(t)j + h(t)K\,dt = \left(\int f(t)\,dt\right)i + \left(\int g(t)\,dt\right)j + \left(\int h(t)\,dt\right)K$$

Ex 1 Page 716 $\int (\cos t\, i + j - 2tK)\,dt$

$= \int \cos t\,dt\, i + \int 1\,dt\, j + \int -2t\,dt\, K$

$= \sin t\, i + t\, j + -t^2 K + C \qquad C = c_1 i + c_2 j + c_3 K$

So, to integrate a vector fun, integrate all components.

Similarly for definite integral.

Ex 2 page 716 $\int_0^{\pi} (\cos t\, i + j - 2tK)\,dt$

$= \sin t\Big|_0^{\pi} i + t\Big|_0^{\pi} j + -t^2\Big|_0^{\pi} K = \pi j - \pi^2 K$

Ex 3 page 716

a glider acceleration vector is $a(t) = -3\cos t\, i - 3\sin t\, j + 2K$

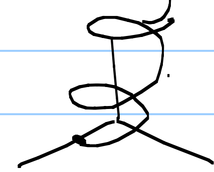
initialy $(t=0)$ possision is $(3,0,0)$ $(r(0) = \langle 3, 0, 0 \rangle)$, and

Velocity is $3j$ $(V(0) = \langle 0, 3, 0 \rangle)$

find the gliders possision function $r(t) = ??$

find $v(t) = \int a(t)\, dt$ then find $r(t) = \int v(t)\, dt$

$r(t) = 3\cos t\, i + 3\sin t\, j + t^2 K$



?? is it a helix

No: the spiral moves up Not linearly

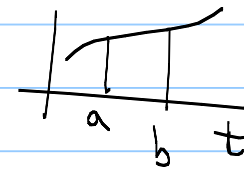
Another App of vector fun $\int$'s is the derivation of Projectile motion under ideal cond-itions

this we will skip. We will see more App in the next section

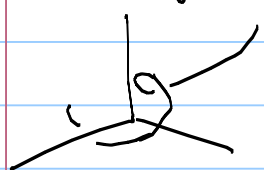
13.3 Arc Length in Space

In a plane the length of the curve definde by $x=f(t)$, $y=g(t)$

from $t=a$ to $t=b$ is $L=\int\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$



In Space when $r(t)=x(t)i+y(t)j+z(t)k$



$$L=\int_{t=a}^{t=b}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$$

but $\frac{dx}{dt}i+\frac{dy}{dt}j+\frac{dz}{dt}k=V(t)$ velocity

$$|V(t)|=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2} \quad \therefore \boxed{L=\int_{t=a}^{t=b}|V(t)|\,dt}$$

We Know $|V(t)|=$ Speed and if speed is constant distance $=$ Speed $*$ time is $L=\int|V(t)|$ consistent with this.

$$L=\int_{t_1}^{t_2}\overbrace{|V|}^{\text{constant}}\,dt=|V|t\Big|_{t_1}^{t_2}=|V|t_2-|V|t_1=\overbrace{|V|(t_2-t_1)}^{\text{speed * time}}$$

Ex 1 page 724 $r(t)=\cos t\,i+\sin t\,j+tk$ a gliders path, helix upwards counter clockwise

find the length of the glider's path from $t=0$ to $t=2\pi$

$$L=\int_0^{2\pi}|V|\,dt=\int_0^{2\pi}\sqrt{(\sin t)^2+(\cos t)^2+1^2}\,dt=2\pi\sqrt{t}\text{ units of length.}$$

Suppose we want the Length from a fixed point $p(t_0)$ called the base point t_0

$t=3$. $L=\int_{t_0}^{3}|v(t)|\,dt$, $t=7$ $L=\int_{t_0}^{7}|v(t)|\,dt$

$t=t$ in general $L=\int_{t_0}^{t}|v(\tau)|\,d\tau$ which is a function of t (scalar)

dummy var

this function is called the arc length parameter with base point $p(t_0)$ and it is denoted by $S(t)$

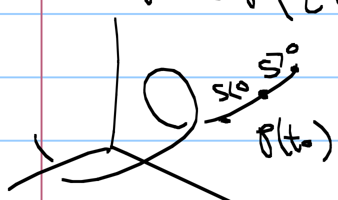
Why it is a curve parameter ??

If $S=f(t)$ we may be able to solve for t in terms of S

$t=t(s)$ and by replacing t with $t(s)$ in $r=r(t)$

We get the curve function in terms of S

$r=r(t(s))$. So tell me the "directed" distance along the curve from the base point the function $r(s)$ gives the point on the curve with that distance



(($s>0$ point is in the direction of motion

$s<0$ = = = = opposite direction))

Not all curves are easy to parametrize as Ex 2. Fortunately we rarely need an exact formula for $S(t)$ or its inverse $t(s)$. However we need the concept for deriving computational formulas.

Ex 2 page 725 Parametrize the curve $r(t) = \cos t\, i + \sin t\, j + t\, k$ with the arc length parameter using the base point $P(t_0 = 0)$

$$s(t) = \int_0^t |v(\tau)|\, d\tau \qquad v = -\sin t\, i + \cos t\, j + 1 \Rightarrow |v| = \sqrt{2}$$

$$= \int_0^t \sqrt{2}\, d\tau = \sqrt{2}\, t$$

So the arc length parameter is $S = \sqrt{2}\, t$

Now solve for $t \Rightarrow t = \frac{S}{\sqrt{2}}$ substitute $t = \frac{S}{\sqrt{2}}$ in $r(t)$

$\Rightarrow r(S) = \cos\frac{S}{\sqrt{2}}\, i + \sin\frac{S}{\sqrt{2}}\, j + \frac{S}{\sqrt{2}}\, k$ which is the parametrization of the curve $r(t)$ with the arc length $\underline{\underline{S}}$

$r(S)$ Identifies a point on the curve with its directed distance from the base point $P(t_0) = (1, 0, 0)$.

Note: The arc length parameter S is an increasing function of t.

$$S(t) = \int_{t_0}^{t} |v(\tau)|\, d\tau$$

by the FTC $\frac{ds}{dt} = |v(t)|$ "Note again that this is consistant with what we know: [illegible]

$\Rightarrow \frac{ds}{dt} > 0$ since speed is nonnegative

$\therefore S$ is increasing function of t

Unit Tangent Vector

If $r = r(t)$ then $V = \frac{dr}{dt}$ is the tangent Vector to the curve $r(t)$ and thus

$T = \frac{V}{|V|}$ is a unit Tangent vector

This is one of three unit vectors in a refference frame that describs the motion of an object traveling in 3D

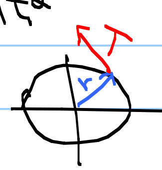
Ex 3 Find the unit Tangent vector of the curve

$$r(t) = 3\cos t\, i + 3\sin t\, j + t^2 K$$

$$V(t) = (-3\sin t)\, i + 3\cos t\, j + 2t K$$

$$T = \frac{V}{|V|} = \frac{-3\sin t}{\sqrt{9+4t^2}}\, i + \frac{3\cos t}{\sqrt{9+4t^2}}\, j + \frac{2t}{\sqrt{9+4t^2}}\, K$$

Ex $r(t) = \cos t\, i + \sin t\, j$ 2D circle



$V = -\sin t\, i + \cos t\, j$ $T = \frac{V}{|V|} = \frac{-\sin t}{1}\, i + \frac{\cos t}{1}\, j = r$

Now show that $\frac{dr}{ds} = T$ page 727

for $r(t)$, $\frac{dr}{dt} = V$ is the change in the position vector r with respect to t, but how about $\frac{dr}{ds}$ (how does the position vector change with respect to the arc length)

Since S is increasing, it has an inverse $t = t(s)$

and $\frac{dt}{ds} = \frac{1}{ds/dt}$ section 7.1 $= \frac{1}{|V|}$

by the Chain Rule $\frac{dr}{ds} = \frac{dr}{dt}\frac{dt}{ds} = V\frac{1}{|V|} = T$

So the Unit Tangent Vector the rate of change in the position vector with respect to the arc length.

Note: if curve is Not $\boxed{\text{smooth}}$ $\left(\frac{dr}{dt} = V = \langle 0,0,0\rangle\right)$ then T is Not defined

13.4 Curvature and Normal Vector of a Curve

In this section, we will study how a curve turns or bends

Curvature of a plane curve

The Magnitude of T is $|T|=1$ constant but its direction changes

Curvature is defined as $K = \left|\frac{dT}{ds}\right|$ the magnitude of the change in T with respect to s

?? which has more K

s is small so K is big

?? What is K for ——— (straight line) $K=0$

To calculate K Note that wee need s.

1- parametrize $r(t)$ with s to get $r(s)$

2- $T = \frac{dr}{ds}$ T is function s

3- $K = \left|\frac{dT}{ds}\right| = \left|\frac{d^2r}{ds^2}\right|$

K is function of s

Too Long

too hard

s not always easy to find

chain rule

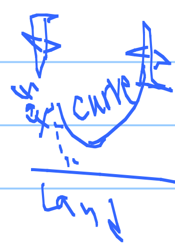
$$K=\left|\frac{dT}{ds}\right|=\left|\frac{dT}{dt}\,\frac{dt}{ds}\right|=\left|\frac{dT}{dt}\,\frac{1}{\frac{ds}{dt}}\right|=\left|\frac{dT}{dt}\,\frac{1}{|V|}\right|$$

So $K=\left|\frac{dT}{dt}\cdot\frac{1}{|V|}\right|$ much easier and No need for parametrization with arc length.

1) $V=\frac{dr}{dt}$, $T=\frac{V}{|V|}$ T is function of t

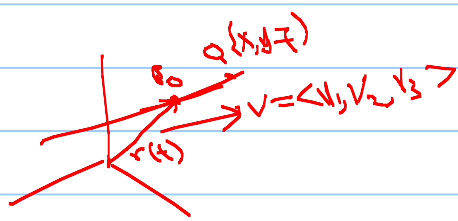
2) $K=\frac{1}{|V|}\left|\frac{dT}{dt}\right|$ K is function of t

Note K and Speed are inversly related



Ex 1 page 729 for straight line

$K=0$



$\vec{PQ}=t\vec{V}$

Scalar parametric form of a line $\Rightarrow$ $x=x_0+tv_1$, $y=y_0+tv_2$, $z=z_0+tv_3$

or $r(t)=\langle x,y,z\rangle = C+tv$ $\quad \langle x_0,y_0,z_0\rangle$

Vector form of a line

$$r(t)=\vec{C}+t\vec{V}$$

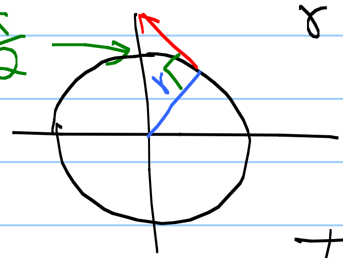
$T=\frac{V}{|V|}$ $\quad V=r'(t)=\vec{V} \Rightarrow T=\frac{\vec{V}}{|\vec{V}|}$

$\frac{dT}{dt}=0$ $\langle 0,0,0\rangle$ $\Rightarrow K=\frac{1}{|\vec{V}|}\cdot\left|\frac{dT}{dt}\right|=0$ as expected.

13.1 page 731

Ex 2 page 729 find the curvature of

$r(t) = a\cos t\, i + a\sin t\, j$ (circle of radius a)



$$K = \frac{1}{|v|}\left|\frac{dT}{dt}\right|$$

$$T = \frac{v}{|v|} \qquad v = \frac{dr}{dt} = -a\sin t\, i + a\cos t\, j$$

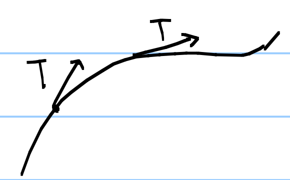
$$|v| = a \Rightarrow T = -\sin t\, i + \cos t\, j \Rightarrow \frac{dT}{dt} = -\cos t\, i - \sin t\, j$$

$$\Rightarrow \left|\frac{dT}{dt}\right| = 1 \Rightarrow K = \frac{1}{a}$$

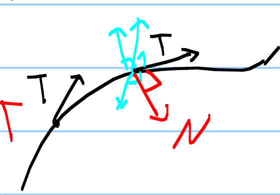
For Fun find K using $K = \left|\frac{dT}{ds}\right|$. in this case it is fairly easy

Note: We can use $K = \left|\frac{dT}{dt} \cdot \frac{1}{|v|}\right|$ for Curves in Space but in the next section we will learn a more convenient formula.

Note: as $T = \frac{v}{|v|} = \frac{r'(t)}{|r'(t)|}$ changes direction, the curve bends. And we define the rate of change of T with respect to S as curvature $K = \left|\frac{dT}{ds}\right| = \frac{1}{|v|}\left|\frac{dT}{dt}\right|$



Another important unit vector is a normal vector to T which is the Normal to T in the direction of the turn.



Unit Normal Vector

We have seen in 13.1 that if $|r(t)| = \text{Constant}$ then $r'(t)\cdot r(t) = 0$. $\therefore \frac{dT}{ds}\cdot T = 0 \quad (|T|=1)$

$r(t)\cdot r(t) = |r(t)|^2 = c^2$

$\frac{d}{dt}(r(t)\cdot r(t)) = 0$

$\Rightarrow \frac{dT}{ds}$ is Normal to T

$r'(t)\cdot r(t) + r(t)\cdot r'(t) = 0$

$2r'(t)\cdot r(t) = 0$

$r'(t)\cdot r(t) = 0$

$\Rightarrow \dfrac{\frac{dT}{ds}}{\left|\frac{dT}{ds}\right|}$ is a Unit Normal to T. but $K = \left|\frac{dT}{ds}\right|$

So the principal unit Normal to T is $\boxed{N = \frac{1}{K}\frac{dT}{ds}}$

Note this formula requires K and s

$$N = \frac{1}{K}\frac{dT}{ds} = \frac{1}{\underbrace{\frac{1}{|v|}\left|\frac{dT}{dt}\right|}_{K}}\,\underbrace{\frac{dT}{dt}\,\frac{dt}{ds}}_{\frac{dT}{ds}} = \frac{1}{\cancel{\frac{1}{|v|}}\left|\frac{dT}{dt}\right|}\,\frac{dT}{dt}\,\overbrace{\frac{1}{\cancel{\frac{ds}{dt}}}}^{\frac{dt}{ds}}$$

$\frac{ds}{dt} = |v|$

$$\Rightarrow \boxed{N = \frac{\frac{dT}{dt}}{\left|\frac{dT}{dt}\right|}}$$

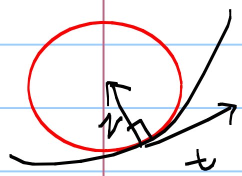
Ex 3 page page 730 Find T and N for the circular motion $r(t) = (\cos 2t)i + (\sin 2t)j$.

$V = -2\sin 2t\, i + 2\cos 2t\, j \Rightarrow T = \frac{-2\sin 2t\, i + 2\cos 2t\, j}{2} = -\sin 2t\, i + \cos 2t\, j$

$\frac{dT}{dt} = -2\cos 2t\, i - 2\sin 2t\, j \Rightarrow N = \frac{-2\cos 2t\, i - 2\sin 2t\, j}{2} = -(\cos 2t)i - (\sin 2t)j$

Circle of Curvature. (osculating circle)

<u>The</u> circle of curvature at a point P is the circle which



1) is tangent to the curve at P
(has same T as the curve at P)

2) has the same curvature as the curve at P

3) lies tward N of the curve at P

The radius of this circle is ?? $\rho = \frac{1}{K}$ {as we saw in Ex 2 page 729}

Ex 4 Page 731

Find and graph the osculating circle of $y = x^2$ at the origion

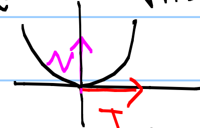
cartesian eqn?? No worry, in section 11.1 we learned how to parametrize a curve easily.

let $x = t \Rightarrow y = t^2$ $\therefore$ the vector representation of the curve is $r(t) = t\,i + t^2 j$

We Need the Normal and the curvature so we need T

$$T = \frac{1i + 2t j}{\sqrt{1^2 + 4t^2}} = \frac{1}{\sqrt{1+4t^2}}\, i + \frac{2t}{\sqrt{1+4t^2}}\, j \quad \text{at } t=0 \text{ (origion) } T = i$$

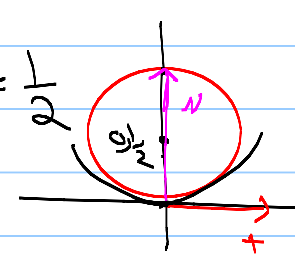
$\therefore N = j$ (you can solve for it)



$$\frac{dT}{dt} = -\frac{1}{2}(1+4t^2)^{-3/2}\, i + \left(2(1+4t^2)^{-\frac{1}{2}} - 8t(1+4t^2)^{-3/2}\right) j$$

$$K_{t=0} = \frac{1}{|v|}\left|\frac{dT}{dt}\right|_{t=0} = \frac{1}{\sqrt{1+4(0)}}\left|(2j)\right| = |2j| = 2 \quad \therefore \rho = \frac{1}{2}$$

$\Rightarrow$ center is $(0, \frac{1}{2}) \Rightarrow$ eqn is $(x-0)^2 + (y - \frac{1}{2})^2 = (\frac{1}{2})^2$



Note: Circle is better App of the curve than tangent.

K & N for Space curves.

Just as for plane curves

$$T = \frac{dr}{ds} = \frac{V}{|V|} = \frac{\frac{dr}{dt}}{\left|\frac{dr}{dt}\right|}$$

$$K = \left|\frac{dT}{ds}\right| = \frac{1}{|V|}\left|\frac{dT}{dt}\right|$$

$$N = \frac{\frac{dT}{ds}}{\left|\frac{dT}{ds}\right|} = \frac{1}{K}\frac{dT}{ds} = \frac{\frac{dT}{dt}}{\left|\frac{dT}{dt}\right|}$$

Ex 5 page. + Ex 6 Find the curvature for the helix
page 732 + 733

$r(t) = (a\cos t)i + (a\sin t)j + btK$

$a, b \geq 0 \quad a^2+b^2 \neq 0 \Rightarrow |v| = 0$ K is not defined

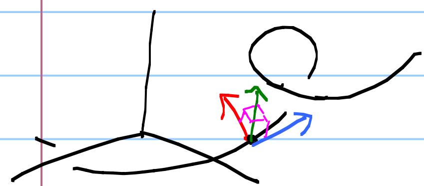
Then analize it based on different values of a and b

See page 732 at the bottom

Then find N for the helix (Ex 6) and describe how the vector is turning

13.5 Tangential and Normal components of Acceleration

Before that, The TNB Frame



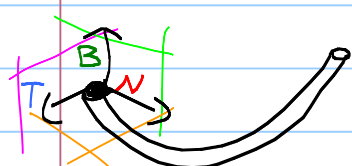
$r(t)$ is the position vector for a moving particle in space. to the particle, the cartesian i, j, and k coordinates are not truly relevant. what is relevant are

1) the particles forward direction (the unit tangent vector T)
2) The particles turning direction (the unit normal vector N)
3) The particles twist direction (the unit binomial vector B)

(The direction of exiting the plane determined by T and N)

Together, These vectors define the particles moving frame which is called The Frenet frame or TNB frame.

silent

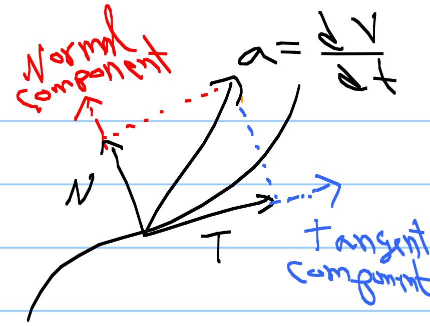


The three planes determined by by T, N, and B are called $\underbrace{\text{osculating}}_{T\ \&\ N}$, $\underbrace{\text{Normal}}_{B\ \&\ N}$, and $\underbrace{\text{rectifying}}_{T\ \&\ B}$

Now use maple Tools, Tutors, vector Calc, Space curve. File, close and return plot.

Exercise 7 Page 738. Find r, T, N, B at $t = \frac{\pi}{4}$
then find the osculating, Normal, and rectifying planes at $t = \frac{\pi}{4}$

Tangential and Normal Components of acceleration



The acceleration $a = \frac{d}{dt}v$ vector always lies in the osculating plane (the T and N plane) as we will see. And we usually want to know how much of it in the direction of T and how much in the direction of N.

We want $a = \underline{\qquad}\, T + \underline{\qquad}\, N$

magnitude of acceleration acting in the direction of T

magnitude of acceleration acting in the direction of N

$$a = \frac{dv}{dt} = \frac{d}{dt}\Big(T \frac{ds}{dt}\Big)$$

direction of v

magnitude of v (speed)

chain rule

or $v = \frac{dr}{dt} = \frac{dr}{ds}\frac{ds}{dt}$

$\Rightarrow v = T\frac{ds}{dt}$

product rule

$$\Rightarrow a = \frac{dT}{dt}\frac{ds}{dt} + T\frac{d^2s}{dt^2} = \underbrace{\frac{dT}{ds}\frac{ds}{dt}}\frac{ds}{dt} + T\frac{d^2s}{dt^2}$$

But $N = \dfrac{\frac{dT}{ds}}{\left|\frac{dT}{ds}\right|} \Rightarrow N = \frac{1}{K}\frac{dT}{ds}$ since $K = \left|\frac{dT}{ds}\right|$ $\Rightarrow \frac{dT}{ds} = KN$

$$\Rightarrow a = K\Big(\frac{ds}{dt}\Big)^2 N + \frac{d^2s}{dt^2}T = \underbrace{K|v|^2}_{\text{Normal scalar component of acceleration}} N + \underbrace{\frac{d}{dt}|v|}_{\text{tangent scalar component of acceleration}} T$$

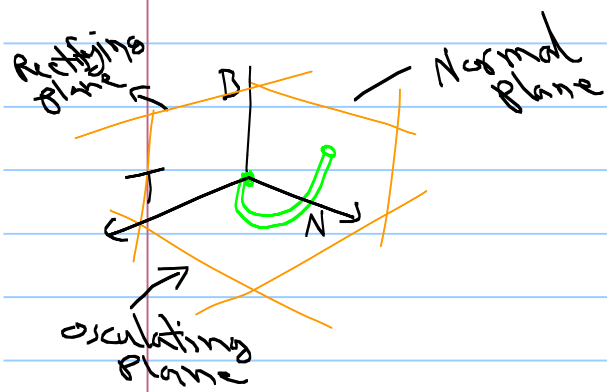
Normal scalar component of acceleration $a_N = \sqrt{|a|^2 - a_T^2}$

tangent scalar component of acceleration a_T

Read the first paragraph after def page 735.

Ex 1 page 736.

Torsion



Curvature $\kappa = \left|\frac{dT}{ds}\right|$ how fast the T changes with respect to s (How fast the Normal plane turns)

Torsion $\tau = -\frac{dB}{ds} \cdot N$ how fast B changes (How fast the osculating plane turns about T)

$$\frac{dB}{ds} = \frac{d(T \times N)}{ds} = \underbrace{\frac{dT}{ds} \times N}_{=0} + T \times \frac{dN}{ds}$$

$= 0$ since $\frac{dT}{ds}$ is in the direction of N $\left(\frac{\frac{dT}{ds}}{\left|\frac{dT}{ds}\right|} = N \right)$

$\Rightarrow \frac{dB}{ds} = T \times \frac{dN}{ds} \Rightarrow \frac{dB}{ds}$ is orthogonal to T

$\Rightarrow \frac{dB}{ds} = -\tau N$ multiple of N (- is convention) this multiple is called Torsion. To solve for it dot both side with N $\Rightarrow \frac{dB}{ds} \cdot N = -\tau$ $\quad (N \cdot N) = |N|^2 = 1$

$\Rightarrow \tau = -\frac{dB}{ds} \cdot N$

Formula to find Torsion

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|v \times a|^2}$$

$v = r'(t)$
$a = v'(t)$
$a' = v''(t) = r'''(t)$

Note formulas Page 756

Exercise 9 Find B and τ for
$r(t) = (3\sin t)i + (3\cos t)j + 4t\,k$

13.5 Exercises

1) $r(t) = a\cos t\, i + a\sin t\, j + bt\, K$

$a_T = \frac{d}{dt}|v|$ $\quad v = -a\sin t\, i + a\cos t\, j + b$

$|v| = \sqrt{a^2+b^2}$

$a_T = \frac{d}{dt}\left(\sqrt{a^2+b^2}\right) = 0$

$a_N = K|v|^2 = \frac{|a|}{a^2+b^2}\left(\sqrt{a^2+b^2}\right)^2 = |a|$

So $a = |a|N + 0T$

5) $r(t) = t^2 i + (t + \frac{1}{3}t^3)j + (t - \frac{1}{3}t^3)K$ $\quad t=0$

$a = a_N N + a_T T$ $\quad a_T = \frac{d}{dt}|v|$ $\quad a_N = K|v|^2 = \sqrt{|a|^2 - a_T^2}$

$v = 2t\, i + (1+t^2)j + (1-t^2)K \Rightarrow |v| = \sqrt{4t^2 + (1+t^2)^2 + (1-t^2)^2}$

$a_T = \frac{d}{dt}|v| = \frac{1}{2}\left(4t^2 + (1+t^2)^2 + (1-t^2)^2\right)^{-\frac{1}{2}}\left(8t + 2(1+t^2)2t + 2(1-t^2)(-2t)\right)$

$a_T(0) = \frac{1}{2}(0+1+1)^{-\frac{1}{2}}(0+0+0) = 0$

$a = 2i + 2tj - 2tK \Rightarrow |a(0)| = \sqrt{4+0-0} = 2$

$\therefore a_n(0) = \sqrt{2^2 - 0^2} = 2$ $\quad \therefore a = 2N + 0T$

7) $r(t) = \cos t\, i + \sin t\, j - K \qquad t = \pi/4$

TNB frame

$V = -\sin t\, i + \cos t\, j - 0K \qquad |V| = 1$

$T = -\sin t\, i + \cos t\, j \Rightarrow T(\frac{\pi}{4}) = \frac{-1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j$

$\frac{dT}{dt} = -\cos t\, i - \sin t\, j \Rightarrow \left|\frac{dT}{dt}\right| = 1$

$\therefore N = -\cos t\, i - \sin t\, j \Rightarrow N(\frac{\pi}{4}) = \frac{-1}{\sqrt{2}} i - \frac{1}{\sqrt{2}} j$

without solving for B from graph B = K

$$B = T \times N = \begin{vmatrix} i & j & K \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{vmatrix} = (0)i - (0)j + (1)K = K$$

9) $r(t) = 3\sin t\, i + 3\cos t\, j + 4t K$

$T = \frac{3}{5}\cos t\, i - \frac{3}{5}\sin t\, j + \frac{4}{5} K \qquad N = -\sin t\, i - \cos t\, j \qquad K = \frac{3}{25}$

from Exersise 9 13.4

$$B = T \times N = \begin{vmatrix} i & j & K \\ \frac{3}{5}\cos t & \frac{-3}{5}\sin t & \frac{4}{5} \\ -\sin t & -\cos t & 0 \end{vmatrix} = \frac{4}{5}\cos t\, i - \frac{4}{5}\sin t\, j - \frac{3}{5} K$$

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|v \times a|^2}$$ might sam time $\quad K = \frac{|v \times a|}{|v|^3} \Rightarrow |v \times a| = K|v|^3$

$$V = 3\cos t\, i - 3\sin t\, j + 4K \quad \Rightarrow |V| = \sqrt{9+16} = 5$$

$$a = -3\sin t\, i - 3\cos t\, j + 0K$$

$$|V \times a| = K|V|^3 = \frac{3}{25} 5^3 = 15$$

$$\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix} = \begin{vmatrix} 3\cos t & -3\sin t & 4 \\ -3\sin t & -3\cos t & 0 \\ -3\cos t & 3\sin t & 0 \end{vmatrix} = 3\cos t(0) - -3\sin t(0) + 4(-9\sin^2 t - 9\sin^2 t)$$

$$= 4(-9) = -36$$

$$\therefore \tau = \frac{4(-9)}{15^2} = \frac{-4}{25}$$

Note path is allways moving down

16) $r(t) = \cosh t\, i - \sin t\, j + t\, K$

$V = \sinh t\, i - \cosh t\, j + 1 K \Rightarrow |V| = \sqrt{\cosh 2t + 1}$

$\Rightarrow T = \frac{1}{\sqrt{2}} \tanh t\, i - \frac{1}{\sqrt{2}} j + \frac{1}{\sqrt{2}} \operatorname{sech} t\, K \qquad = \sqrt{2\cosh^2 t}$

$= \sqrt{2} \cosh t$

$\cosh 2t = \cosh^2 + \sinh^2$

$\cosh^2 t = \frac{\cosh 2t + 1}{2}$

$\frac{dT}{dt} = \frac{1}{\sqrt{2}} \operatorname{sech}^2 t\, i - 0 j - \frac{1}{\sqrt{2}} \operatorname{sech} t \tanh t\, K$

$\left|\frac{dT}{dt}\right| = \sqrt{\frac{1}{2}\operatorname{sech}^4 t + \frac{1}{2}\operatorname{sech}^2 t \tanh^2 t}$

$= \frac{1}{\sqrt{2}} \operatorname{sech} t \sqrt{\underbrace{\operatorname{sech}^2 t + \tanh^2 t}_{=1}} = \frac{1}{\sqrt{2}} \operatorname{sech} t$

$\therefore N = \operatorname{sech} t\, i - \tanh t\, K$

$\therefore B = \begin{vmatrix} i & j & K \\ \frac{1}{\sqrt{2}}\tanh t & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\operatorname{sech} t \\ \operatorname{sech} t & 0 & -\tanh t \end{vmatrix} = \frac{1}{\sqrt{2}} \operatorname{sech} t\, i - \left(-\frac{1}{\sqrt{2}}\tanh^2 t - \frac{1}{\sqrt{2}}\operatorname{sech}^2 t\right) j$

$+ \frac{1}{\sqrt{2}} \operatorname{sech} t\, K$

$\Rightarrow B = \frac{1}{\sqrt{2}} \operatorname{sech} t\, i + \frac{1}{\sqrt{2}} j + \frac{1}{\sqrt{2}} \operatorname{sech} t\, K$

for Torsion $a = \cosh t\, i - \sinh t\, j + 0 K$

$\cosh^2 - \sinh^2 = 1$

$V \times a = \begin{vmatrix} i & j & K \\ \sinh t & -\cosh t & 1 \\ \cosh t & -\sinh t & 0 \end{vmatrix} = +\sinh t\, i - -\cosh t\, j + \left(-\sinh^2 t - -\cosh^2 t\right) K$

$= \sinh t\, i + \cosh t\, j + 1 K$

$$|v \times a|^2 = \left(\sqrt{\sinh^2 t + \cosh^2 t + 1}\right)^2 = \sinh^2 t + \cosh^2 t + 1$$

$$\therefore \quad \tau = \frac{\begin{vmatrix} \sinh t & -\cosh t & 1 \\ \cosh t & -\sinh t & 0 \\ \sinh t & -\cosh t & 0 \end{vmatrix}}{|v \times a|^2} = \frac{0 - 0 + 1(-\cosh^2 t - -\sinh^2 t)}{|v \times a|^2}$$

$$\tau = \frac{1}{\sinh^2 t + \cosh^2 t + 1} = \frac{1}{2\cosh^2 t}$$

$\cosh^2 t$

Ch 14 Partial Derivatives

In a single variable function, $y=f(x)$, where there is only one independent variable, the rate of change of y (the dependent) solely depends on the change of x

However, many functions depend on more than one variable such as $V=\pi r^2 h$ (the volume of a cylinder). In these functions the changes of the dependent with respect to the independents are more varied and interesting than functions of one variable
derivatives

14.1 Functions of several variables

Definition:

If D is a set of n-tuples real numbers $(x_1, x_2, \dots, x_n)$ Then a real-valued function on D is a rule That assigns a unique real number $w=f(x_1, x_2, \dots, x_n)$ to each element in D

$x_1, x_2, \dots, x_n$ are independent variables. W is the dependent

Examples of functions $\quad y=f(x)$ single ind var

Note (these are the convention Letters for ind and dep variables) $\to$ $z=f(x,y)$ Two ind var

$w=f(x,y,z)$ Three ind var

For more than three $D=f(x_1, x_2, x_3, x_4)$

When doing App we use letters that describe what the variables stand for. $V=\pi r^2 h$

Domain and Ranges of fun of several vars

as in the case of a single var fun, if the domain is not specified, then it will be the set of n-tuples $(x_1, x_2, \ldots, x_n)$ that does not lead to complex numbers or division by zero (leads to real numbers)

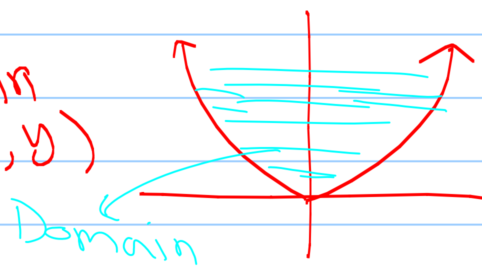
Ex 1 page 748

a) $Z = \sqrt{y - x^2}$

	Domain	Range
	$y \geq x^2$	$[0, \infty)$

How do I know for sure $z \in [0, \infty)$
Fix $x = 0$, $z = \sqrt{y}$, $y \geq 0$
will give $z \in [0, \infty)$

Note that the points in the domain are pairs of real numbers (x, y)
D is a region in the xy-plane such that $y \geq x^2$

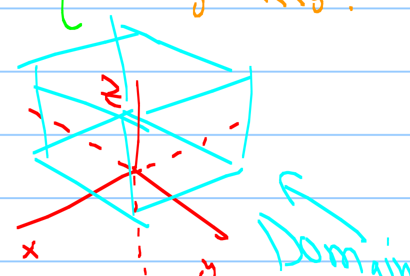


b) $W = xy \ln Z$

	Domain	Range
	$Z > 0$ x, y real numbers	$(-\infty, \infty)$

How do I know for sure $W \in (-\infty, \infty)$
fix $z = e$, $y = 1$
$W = x$, x any thing, so
W any thing.

Note that the points in the domain are triplets of real numbers (x, y, z)
D is a region in space where $Z > 0$ (the half space above the xy-plane)

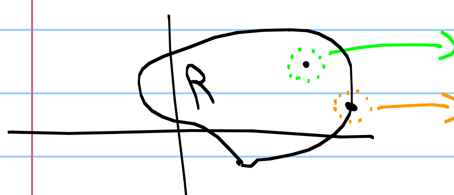


Funstions of Two variables (we mean independent variables)

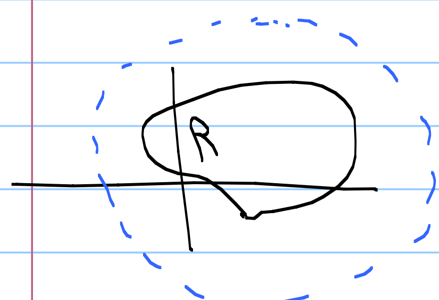
For a function of two variables $z=f(x,y)$, The Domain is a region in the xy-plane

Just as in $y=f(x)$ the domain is an interval that is either Closed, open, or neither ($[a,b]$, (a,b), $(a,b]$)

The domain of $z=f(x,y)$ is a region that is either closed, open, or neither.



See definitions page 749



See definitions Page 749

Ex 2 page 749 Describe the domain of $z=\sqrt{y-x^2}$

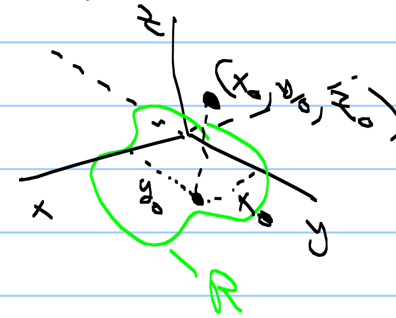
D $y \geq x^2$ all boundry points are included
$\Rightarrow$ closed region

The region does not lie in a disk of fixed radius $\Rightarrow$ unbounded region

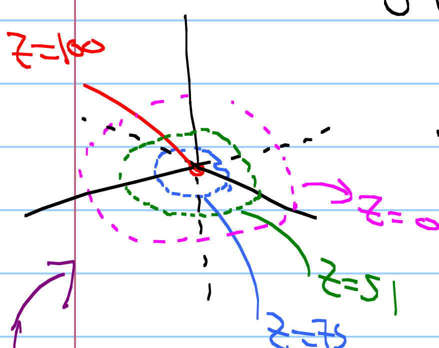
Graphs, level curves, and Contures of $z=f(x,y)$

The graphs of $z=f(x,y)$ are the set of points (x,y,z) in space which are called <u>surfaces.</u>

The domain is R in xy-plane
The surface consists of points (x_0,y_0,z_0) that are verticaly away from (x_0,y_0) a directed distance $z_0=f(x_0,y_0)$



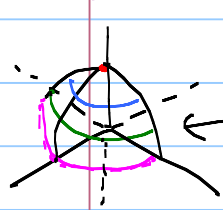
Ex 3 Page 750 graph $z=100-x^2-y^2$ Note D is the xy plane



we don't want to plot all points !!

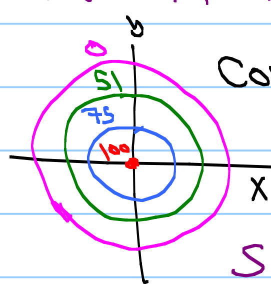
but if $z=0 \Rightarrow 0=100-x^2-y^2 \Rightarrow x^2+y^2=100$
if $z=51 \Rightarrow x^2+y^2=49$
if $z=75 \Rightarrow x^2+y^2=25$
if $z=100 \Rightarrow x^2+y^2=0$

these curves (on the xy plane) are called <u>Level curves</u> ($z=c$)
The curves on the surface with fixed z values are <u>conture curves</u>



conture curves on the surface

Paraboloid 12.6



Conture graph (map) (graph of Level curve)
(projection of contur curves on xy-plane)

See Figure 14.7 page 751

Note paragraph bellow Figure 14.7 page 751

Functions of Three Variables

Comparison

	$y=f(x)$	$z=f(x,y)$	$w=f(x,y,z)$
# of ind var	1	2	3
Domain	interval in a Line (1D)	Region in a Plane (2D)	Region in Space (3D)
graph	curve in 2D	surface in 3D	in 4D can't imagine
Levels	No	level curve in 2D	Level Surface in 3D

The set of points (x,y,z) in space where $f(x,y,z)=c$ is called a level surface.

Ex 4 page 750 Describe the level surface of $f(x,y,z)=\sqrt{x^2+y^2+z^2}$

function of 3 ind var $\therefore$ domain region in space (3D)
Graph in 4D can't imagine

Level surfaces are $c=\sqrt{x^2+y^2+z^2} \Rightarrow c^2=x^2+y^2+z^2$
which are spheres in 3D

For any point on a specific sphere the value of the function is constant
as we move in or out to another sphere the value of the function changes

☆ you may want to use Zeghos examples on level curves.

14.1 Exersises

1) $f(x,y)=x^2+xy^3$

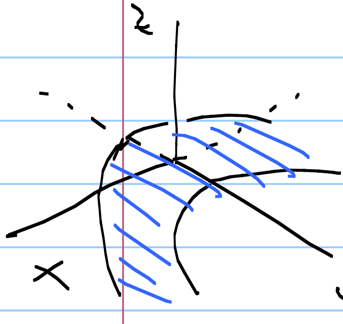
a) $f(0,0)=0^2+(0)(0)^3=0$

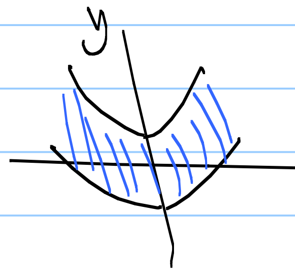
b) $f(-1,1)=(-1)^2+-1(1)^3=1-1=0$

⋮

9) $f(x,y)=\cos^{-1}(y-x^2)$ domain of $\cos^{-1}(\theta)$ is $[-1,1]$

$\therefore$ Domain of $\cos^{-1}(y-x^2)$ is $-1\leq y-x^2\leq 1$





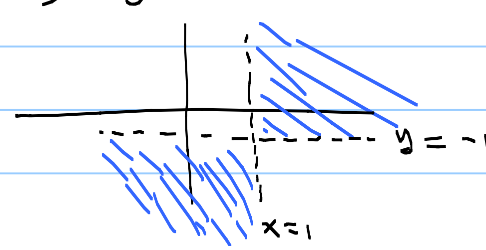
$x^2-1\leq y\leq x^2+1$

10) $f(x,y)=\ln(xy+x-y-1)$

Domain $xy+x-y-1>0 \Rightarrow y(x-1)>(1-x)$

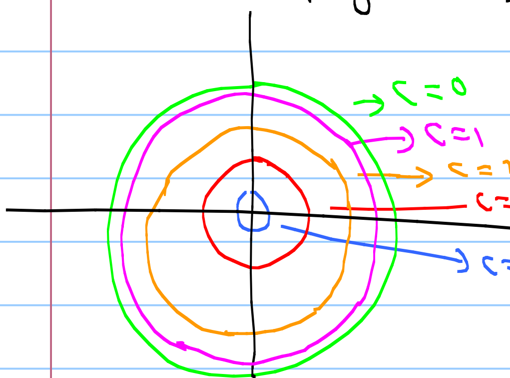
$y>\frac{(1-x)}{(x-1)}$ if $x-1>0$, $x>1 \Rightarrow y>-1$ $x>1$

$y<\left(\frac{1-x}{x-1}\right)$ if $x-1<0$, $x<1 \Rightarrow y<-1$ $x<1$



16) $f(x,y)=\sqrt{25-x^2-y^2}$ $\quad c=0,1,2,3,4$

$c=\sqrt{25-x^2-y^2} \Rightarrow c^2=25-x^2-y^2 \Rightarrow x^2+y^2=25-c^2$



$\to c=0$ circle $r=5$

$\to c=1$ " $r=\sqrt{24}=4.899$

$\to c=2$ " $r=\sqrt{21}=4.58$

$c=3$ " $r=4$

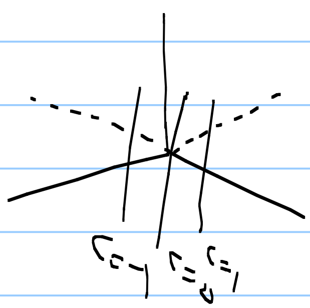
$\to c=4$ " $r=3$

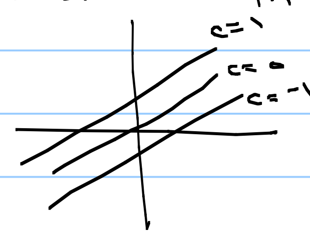
17) $f(x)=y-x$ (Plane through the origion)

a) Domain is the entire xy-plane b) Range $(-\infty,\infty)$

c) Level curves $c=y-x \Rightarrow y=x+c$

Level curves are straight lines with slope equal 1





d) No boundries (domain is the entire xy-plane)

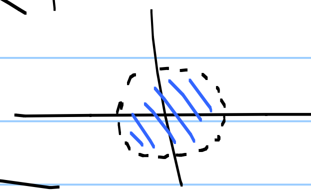
e) Both !

f) unbounded

23) $f(x,y) = \dfrac{1}{\sqrt{16-x^2-y^2}}$

a) Domain $16-x^2-y^2 > 0$ $\quad x^2+y^2<4^2$

all points inside the circle $x^2+y^2=4^2$



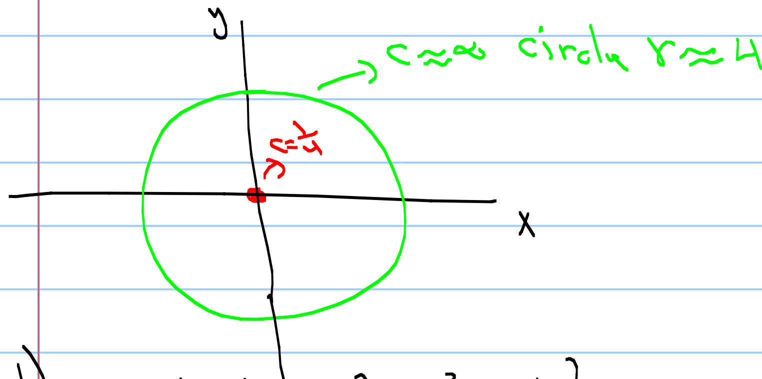
b) the largest the denominator $\sqrt{16-x^2-y^2} = \sqrt{16-(x^2+y^2)}$
when x^2+y^2 is smallest (0)

$\therefore$ the smallest z is $\frac{1}{\sqrt{16}} = \frac{1}{4}$ as x^2+y^2 gets larger to 16
z becomes larger to ∞

$\therefore$ Range is $[\frac{1}{4}, \infty)$

c) $c = \dfrac{1}{\sqrt{16-x^2-y^2}} \Rightarrow \dfrac{1}{c^2} = 16-x^2-y^2 \Rightarrow x^2+y^2 = 16-\dfrac{1}{c^2}$

Level curves are circle centered at the origion



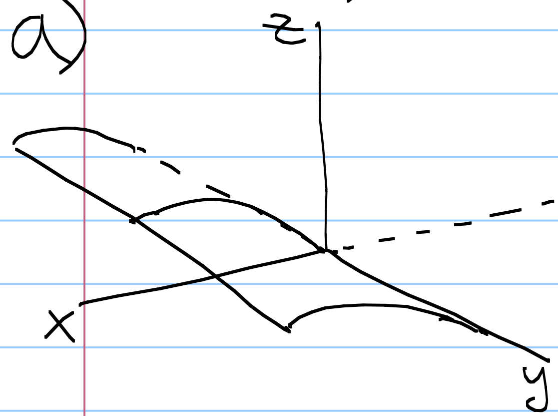
d) the circle $x^2+y^2=4^2$

e) open

f) bounded

38) $f(x,y) = \sqrt{x}$ $\qquad z = \sqrt{x}$ (cylinder)

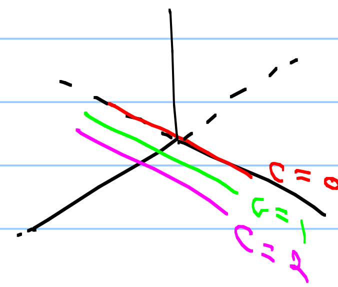
a)



b) Level curves $c = \sqrt{x}$

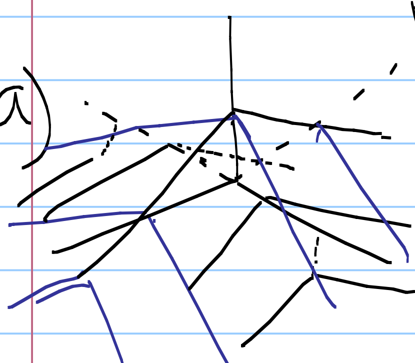
$\Rightarrow x = c^2$

Note $c \geq 0$



46) $f(x,y) = 1 - |x| - |y|$

a)



if

$y = 0 \Rightarrow z = 1 - |x|$

$y = \pm 1 \Rightarrow z = -|x|$

$y = \pm 2 \Rightarrow z = -|x| - 1$

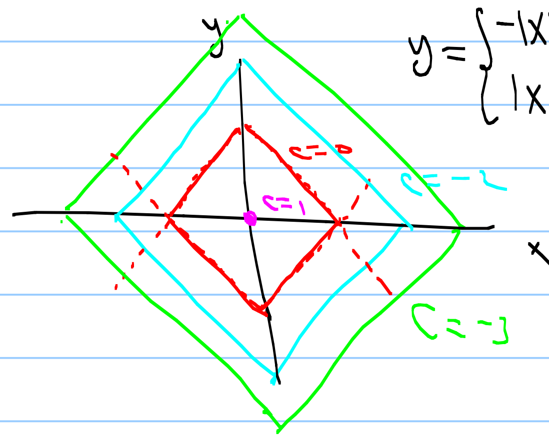
similarly for x

b) $c = 1 - |x| - |y|$

$|y| = 1 - |x| - c$

$|y| = -|x| + 1 - c$

$$y = \begin{cases} -|x| + 1 - c & y \geq 0 \\ |x| - 1 + c & y < 0 \end{cases}$$



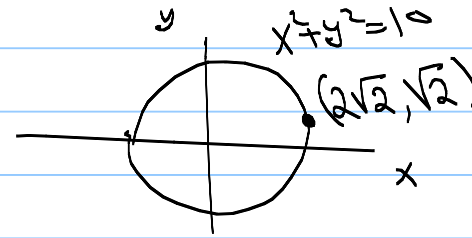
c=0
c=1 Largest z
c=-1
c=-2

49) $f(x,y) = 16 - x^2 - y^2 \qquad (2\sqrt{2}, \sqrt{2})$

$$c = 16 - x^2 - y^2 \Rightarrow c = 16 - (2\sqrt{2})^2 - (\sqrt{2})^2$$
$$c = 6$$

$6 = 16 - x^2 - y^2$

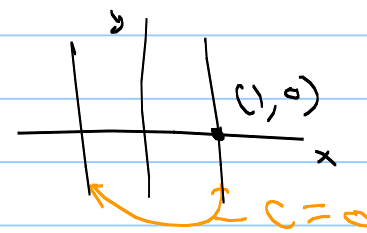
$\therefore\ x^2 + y^2 = 10$ is the level curve through $(2\sqrt{2}, \sqrt{2})$



50) $f(x,y) = \sqrt{x^2 - 1}, \quad (1,0) \quad c = \sqrt{x^2-1} \Rightarrow c = \sqrt{1^2 - 1} = 0$

$c = 0 \Rightarrow 0 = \sqrt{x^2 - 1}$

$\Rightarrow x^2 - 1 = 0 \Rightarrow x^2 = 1 \qquad x = \pm 1$



64) $g(x,y,z) = \dfrac{x-y+z}{2x+y-z} \quad (1,0,-2)$

$$c = \frac{x-y+z}{2x+y-z} \Rightarrow c = \frac{1-0+-2}{2(1)+0--2} = \frac{-1}{4}$$

level surface $c = \frac{-3}{4}$ is $\quad \dfrac{-1}{4} = \dfrac{x-y+z}{2x+y-z} \Rightarrow -2x - y + z = 4x - 4y + 4z$

$\Rightarrow -6x + 3y - 3z = 0$ (you can't divide by -6) plane in space. The value of $g(x,y,z)$ at any point in this plane is $\frac{-1}{4}$

This plane contains $(1,0,-2)$

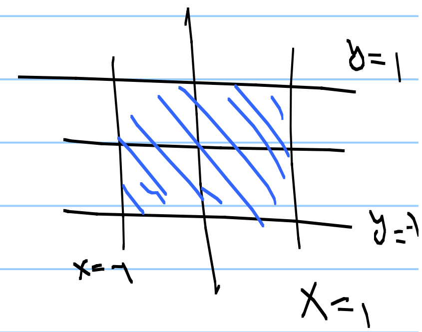
(subset of the Domain)

67) $f(x,y) = \int_x^y \frac{d\theta}{\sqrt{1-\theta^2}}$ $(0,1)$

$f(x,y) = \sin^{-1}(\theta) \Big|_x^y$

$f(x,y) = \sin^{-1}(y) - \sin^{-1}(x)$

Domain $-1 \le y \le 1 \cap -1 \le x \le 1$



for level curve through $(0,1)$

$c = \sin^{-1}(y) - \sin^{-1}(x) \Rightarrow c = \sin^{-1}(1) - \sin^{-1}(0)$

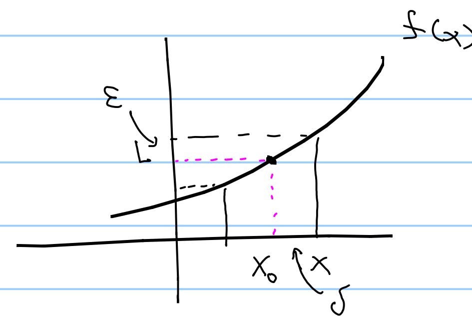
$c = \frac{\pi}{2} - 0$

$c = \frac{\pi}{2}$

∴ level curve through $(0,1)$ is $\frac{\pi}{2} = \sin^{-1}(y) - \sin^{-1}(x)$

14.2 Limits and Continuity in higher dimensions

In one variable fun $y = f(x)$

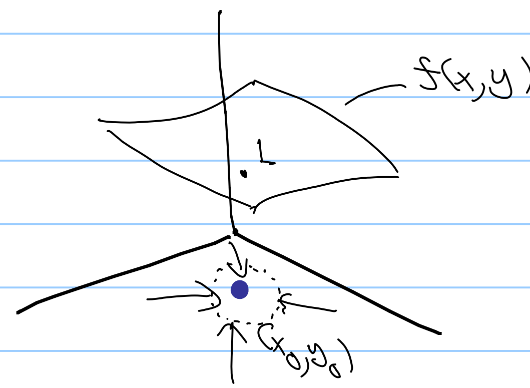


$\lim_{x \to x_0} f(x) = L$ if for every $\varepsilon > 0$

there exists a $\delta > 0$ such that

$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$

In two variable function



$\lim_{(x,y) \to (x_0,y_0)} f(x,y) = L$ if for every $\varepsilon >$

there exists a $\delta > 0$ such that

$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta \Rightarrow |f(x,y) - L| < \varepsilon$

Note Theorem 1 page 757

Ex 1 page 757

(a) $$\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = \frac{0 - (0)(1) + 3}{(0)^2(1) + 5(0)(1) - (1)^3} = -3$$

(b) $$\lim_{(x,y)\to(3,-4)} \sqrt{x^2 + y^2} = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5$$

Ex 2 Page 757

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}.$$ (rewrite)

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x}-\sqrt{y}} = \frac{0}{0} = \lim_{(x,y)\to(0,0)} \frac{x(x-y)}{\sqrt{x}-\sqrt{y}} \cdot \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{x\cancel{(x-y)}(\sqrt{x}+\sqrt{y})}{\cancel{x-y}}$$

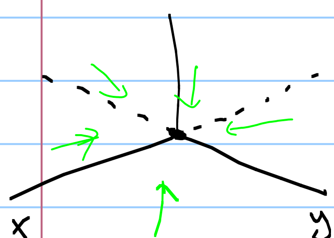
$$= \lim_{(x,y)\to(0,0)} x(\sqrt{x}+\sqrt{y}) = 0$$

Usually Limits of $f(x,y)$ are not easy to find. In some cases (Ex 3) we can use the definition, but using the def is also hard. In other cases we can show that the limit DNE by examining two paths.

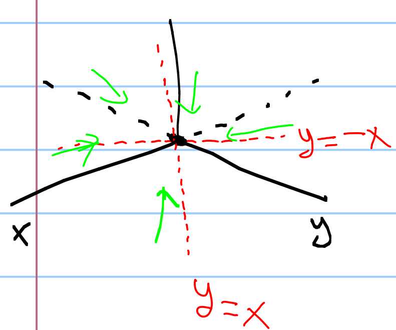
Ex 4 page 759

$$z = f(x,y) = \frac{y}{x} \qquad \lim_{(x,y)\to(0,0)} \frac{y}{x} = \frac{0}{0}$$

can't tell



We can approach $(0,0)$ allong many direction.
If we find two direction where the limit is not the same, then it DNE



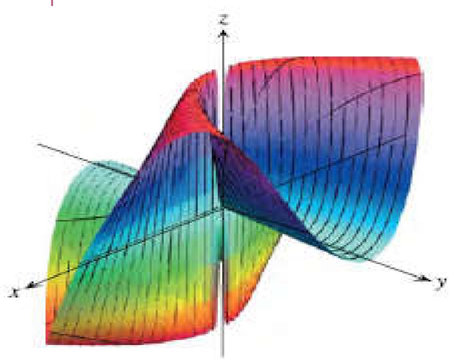
Along $y=x$ $\quad \lim_{(x,y)\to(0,0)} \frac{y}{x} = \lim_{(x,y)\to(0,0)} \frac{x}{x} = 1$

Along $y=-x$ $\quad \lim_{(x,y)\to(0,0)} \frac{y}{x} = \lim_{(x,y)\to(0,0)} \frac{-x}{x} = -1$

$\therefore \lim_{(x,y)\to(0,0)} \frac{y}{x}$ DNE since the limits along different paths are different

Two-Path Test for Nonexistence of a Limit
If a function $f(x, y)$ has different limits along two different paths in the domain of f as (x, y) approaches (x_0, y_0), then $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist.

Ex 6 page 760



EXAMPLE 6 Show that the function

$$f(x, y) = \frac{2x^2y}{x^4 + y^2}$$

(Figure 14.14) has no limit as (x, y) approaches $(0, 0)$.

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} = \frac{0}{0}$$

We want to find two paths in the domain when the limit is different.

No general Method. Try first along $x=x_0$ & $y=y_0$

However, Note, along $y=x^2$ we get a Limit (Not $\frac{0}{0}$)
Note $y=x^2$ is a path to $(0,0)$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x^2}} \frac{2x^2y}{x^4+y^2} = \lim_{(x,y)\to(0,0)} \frac{2x^2x^2}{x^4+x^4} = \lim_{(x,y)\to(0,0)} 1 = 1$$

Now it is easy to see that along $y=ax^2$ will give different limit

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=3x^2}} \frac{2x^2y}{x^4+y^2} = \lim_{(x,y)\to(0,0)} \frac{6x^4}{10x^4} = \frac{6}{10} \neq 1 \therefore \text{DNE}$$

Do Exercises 42, 45, 50 Then $\lim_{P\to(0,0)} \frac{x^2y}{x^2+y^2}$ Polar

Exercise 50 $\lim_{(x,y)\to(1,-1)} \dfrac{xy+1}{x^2-y^2}$ $\dfrac{0}{0}$ $\dfrac{xy+1}{(x-y)(x+y)}$

Looks like along $x=1$

along $y=-x \Rightarrow \dfrac{-x^2+1}{x^2-x^2} = \dfrac{-x^2+1}{0} \Rightarrow \dfrac{0}{0}$ No good

you can't the path is not in the domain

along $x=1 \Rightarrow \dfrac{y+1}{1-y^2} = \dfrac{\cancel{y+1}}{(1-y)(1+y)} \Rightarrow \lim = \dfrac{1}{2}$

along $y=-1 \Rightarrow \dfrac{-x+1}{x^2-1} = \dfrac{-\cancel{(x-1)}}{\cancel{(x-1)}(x+1)} = \dfrac{-1}{x+1} = \dfrac{-1}{2}$ DNE

Along $y=x-1$ & $y=x+1$

Continuity

DEFINITION A function $f(x, y)$ is **continuous at the point (x_0, y_0)** if

1. f is defined at (x_0, y_0),
2. $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists,
3. $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

Ex 5 Page 759

EXAMPLE 5 Show that

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is continuous at every point except the origin (Figure 14.13).

Lim along $y=ax$ for $(0,0)$

gives Lim DNE so Not continuous at $(0,0)$

Continuity of Composites: if $f(x,y)$ is contin and $g(u)$ is cont then $g(f(x,y))$ is continuous

$g(u)=e^u$ $f(x,y)=x-y$

Cont single var continuous

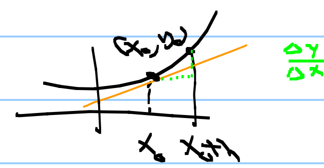
$\Rightarrow g(f) = e^{x-y}$ is continuous

Read Fun of More than Two Vars & Extreme Values Page 761

14.3 Partial derivatives

In single variable fun $y = f(x)$

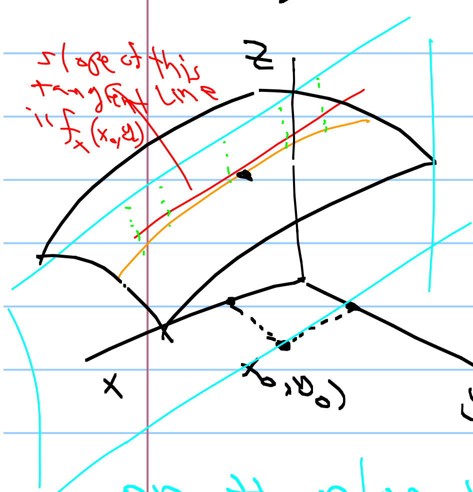
$$\left.\frac{dy}{dx}\right|_{x_0} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$



which is the rate of change of the dependent $\underline{\underline{y}}$ with respect to the independent $\underline{\underline{x}}$

In $z = f(x,y)$, The rate of change in z depends on Two independent variables (x and y).

However, if we fix one of them, we can find the rate of change of z with the other



Definition

$$\lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h} = \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$$

(Partial derivative of f with respect to x)

$\frac{\partial f}{\partial y}$ is similar

On the plane $y = y_0$ $\quad z = f(x)$

Notations. $\frac{\partial f}{\partial x} = f_x = \frac{\partial z}{\partial x} = z_x$

To find partial derivatives, we use the the rules for single variable function since we are keeping only <u>one</u> variable unfixed

Examples 1, 2, 3, 4 implicit, 5 Pages (766-768)

Partial derivatives for funs of more than two var are similar

Example 6 page 768

Second order partial derivatives (4 of them)

$$\frac{\partial^2 f}{\partial x^2} = f_{xx}, \qquad \frac{\partial^2 f}{\partial y \partial x} = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial^2 f}{\partial y^2} = f_{yy}, \qquad \frac{\partial^2 f}{\partial x \partial y} = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

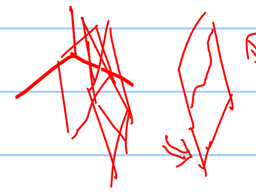
Example 9 page 770 Theorem 2 page 770

Example 10 page 7

Higher order partial derivatives Ex 11 page 7

differentiability

Note: it is not enough that $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ exist for f to be differentiable at (x_0,y_0)



Definition: $z=f(x,y)$ is differentiable at (x_0,y_0) if $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ exist and $\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)$ satisfies $\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

where ε_1 & $\varepsilon_2 \to 0$ as Δx & $\Delta y \to 0$

Theorem 3 page 771

THEOREM 3—The Increment Theorem for Functions of Two Variables Suppose that the first partial derivatives of $f(x, y)$ are defined throughout an open region R containing the point (x_0, y_0) and that f_x and f_y are continuous at (x_0, y_0). Then the change

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

in the value of f that results from moving from (x_0, y_0) to another point $(x_0 + \Delta x, y_0 + \Delta y)$ in R satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

in which each of $\epsilon_1, \epsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$.

COROLLARY OF THEOREM 3 If the partial derivatives f_x and f_y of a function $f(x, y)$ are continuous throughout an open region R, then f is differentiable at every point of R.

for single var $y=f(x)$

$\Delta y = f(x_0+\Delta x) - f(x_0)$ Δx

$\Downarrow$ $\Delta y = f'(x_0)\Delta x + \varepsilon \Delta x$

$\Delta y \approx f'(x)\Delta x$ if $\Delta x \to 0$

$dy = f'(x)\,dx$

$\Delta y \approx dy$

THEOREM 4—Differentiability Implies Continuity If a function $f(x, y)$ is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

Note Ex 8 page 769

Exercise 91 page 775

if $f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & (x,y) \neq 0 \\ 0 & (x,y) = (0,0) \end{cases}$

Show $f_x(0,0)$ & $f_y(0,0)$ exist but f is not differentiable at $(0,0)$ if we show it is Not continuous then it is not differentiable

We use the def of f_x and f_y since the function is defined differently around $(0,0)$.

$$f_x\Big|_{(0,0)} = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{f(h,0) - 0}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0 \qquad \frac{h0^2}{h^2+0} = 0$$

$$f_y\Big|_{(0,0)} = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{f(0,h) - 0}{h} = 0$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{(x,y)\to(0,0)} \frac{ay^4}{(a^2+1)y^4} = \frac{a}{a^2+1}$$

along $x = ay^2$

differ for different a so limit DNE

$\therefore$ Not continuous at $(0,0) \Rightarrow$ Not differentiable at $(0,0)$ Theorem 4

even though $f_x(0,0)$ & $f_y(0,0)$ exist

14.3 Exercises

Note Title 3/4/2018

11) $f(x,y) = \dfrac{x+y}{xy-1}$

$$\frac{\partial f}{\partial x} = \frac{1(xy-1) - y(x+y)}{(xy-1)^2} \qquad \frac{\partial f}{\partial y} = \frac{1(xy+1) - x(x+y)}{(xy-1)^2}$$

19) $f(x,y) = x^y$

$$\frac{\partial f}{\partial x} = y x^{y-1} \qquad \frac{\partial f}{\partial y} = x^y \ln x$$

26) $f(x,y,z) = (x^2+y^2+z^2)^{-\frac{1}{2}}$

$$f_x = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\, 2x$$

$f_y =$ ―――――― y by cymetry

$f_z =$ ―――――― z

3) $f(x,y,z) = yz \ln xy$

$$f_x = yz\,\frac{y}{xy} \qquad f_y = 1z\ln xy + y\left(z\,\frac{x}{xy}\right)$$

$$f_z = y \ln xy$$

34) $f(x,y,z) = \sinh(xy - z^2)$

$f_x = \cosh(xy - z^2)\, y$

$f_y = \cosh(xy - z^2)\, x$

$f_z = \cosh(xy - z^2)\, 2z$

46) $s(x,y) = \tan^{-1} \frac{y}{x}$

$s_x = \frac{1}{1+(\frac{y}{x})^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2} = -y(x^2+y^2)^{-1}$

$s_y = \frac{1}{1+(\frac{y}{x})^2}\,\frac{1}{x} = \frac{1}{x+\frac{y^2}{x}} = \frac{x}{x^2+y^2} = x(x^2+y^2)^{-1}$

$s_{xx} = y(x^2+y^2)^{-2}\, 2x$

$s_{yy} = -x(x^2+y^2)^{-2}\, 2y$

$s_{xy} = -1(x^2+y^2)^{-1} + -y\left(-1(x^2+y^2)^{-2}\, 2y\right)$

$s_{yx} = 1(x^2+y^2)^{-1} + x\left(-1(x^2+y^2)^{-2}\, 2x\right)$

$$S_{xy} = \frac{-1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2} = \frac{-1(x^2+y^2)+2y^2}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

Same

$$S_{yx} = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y)^2}$$

54) $w = x\sin y + y\sin x + xy$

$w_x = \sin y + y\cos x + y$

$w_{xy} = \cos y + \cos x + 1$

$w_y = x\cos y + \sin x + x$

$w_{yx} = \cos y + \cos x + 1$

Same

58) $f(x,y) = 4+2x-3y-xy^2$ $\quad \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ at $(-2,1)$

$$\left.\frac{\partial f}{\partial x}\right|_{(-2,1)} = \lim_{h\to 0} \frac{f(-2+h,1) - f(-2,1)}{h}$$

$$= \lim_{h\to 0} \frac{4+2(-2+h)-3(1)-(-2+h)(1)^2 - [4+2(-2)-3(1)- -2(1)^2]}{h}$$

$$= \lim_{h\to 0} \frac{-3+2h+2-h+1}{h} = 1$$

Check $\frac{\partial f}{\partial x} = 2-y^2 \Rightarrow \left.\frac{\partial f}{\partial x}\right|_{(-2,1)} = 2-1 = 1$ ✓

$$\left.\frac{\partial f}{\partial y}\right|_{(-2,1)} = \lim_{h\to 0} \frac{f(-2,1+h) - f(-2,1)}{h}$$

$$= \lim_{h\to 0} \frac{4+2(-2)-3(1+h)- -2(1+h)^2 - -1}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{-3}\boxed{-3h}+\cancel{2}+4h+2h^2\cancel{+1}}{h} = 1$$

$f_y = -3-2xy \quad f_y(-2,1) = -3-2(-2)1 = 1$ ✓

61 $f(x,y) = 2x+3y+4$ $\quad (2,-1)$

a)

$\frac{\partial f}{\partial y} = 3 \qquad \left.\frac{\partial f}{\partial y}\right|_{(2,-1)} = 3$

b)

$\frac{\partial f}{\partial x} = 2 \qquad \left.\frac{\partial f}{\partial y}\right|_{(2,-1)} = 2$

66) $\underbrace{x}_{f(y,z)}\, z + y\ln x - x^2 + 4 = 0 \qquad x = f(y,z)$

hard to solve for x.

$$\underbrace{\frac{\partial x}{\partial z} z + x} + y\,\frac{\frac{\partial x}{\partial z}}{x} - 2x\frac{\partial x}{\partial z} + 0 = 0$$

$$\frac{\partial x}{\partial z} = \frac{-x}{\left(z + \frac{y}{x} - 2x\right)} \Rightarrow \left.\frac{\partial x}{\partial z}\right|_{(1,-1,-3)} = \frac{-1}{\left(-3 + \frac{-1}{1} - 2(1)\right)} = \frac{1}{6}$$

75) $f(x,y) = e^{-2y} \cos 2x$

$\frac{\partial f}{\partial x} = e^{-2y} \cos 2x \, 2 \qquad \frac{\partial f}{\partial y} = e^{-2y}(-2) \cos 2x$

$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 0$

91) $f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

$f_x = \frac{y^2(x^2+y^4) - 2x(xy^2)}{(x^2+y^4)^2}$ This is partial x for $(x,y) \neq (0,0)$

Since f is defined differently around (0,0) we need to use the definition.

$f_x\big|_{(0,0)} = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$

$f_y\big|_{(0,0)} = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = 0$

$\lim_{\substack{P\to(0,0)\\ \text{along } x=ay^2}} \frac{ay^4}{(a^2+1)y^4} = \frac{a}{a^2+1}$ which is different for different paths

$\therefore$ Lim DNE

$\Rightarrow$ Not continuous at $(0,0) \Rightarrow$ Not differentiable even though f_x, f_y exist at $(0,0)$

14.4 The Chain Rule

Note Title 2/26/2018

If $w = f(x)$ $x = f(t)$, then

$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}$$ Chain rule for single variable.

$$w = f(x(t)) = \frac{dw}{dt} = f'(x)\,x'(t)$$

for functions of several variables the chain rule works the same but it has many forms depending on the variables involved

If $w = f(x,y)$ $x = x(t)$ $y = y(t)$ is differentiable

Then $\Delta w = f_x \Delta x + f_y \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

ε_1 & $\varepsilon_2 \to 0$ as Δx & $\Delta y \to 0$

$$\Rightarrow \frac{\Delta w}{\Delta t} = f_x \frac{\Delta x}{\Delta t} + f_y \frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}$$

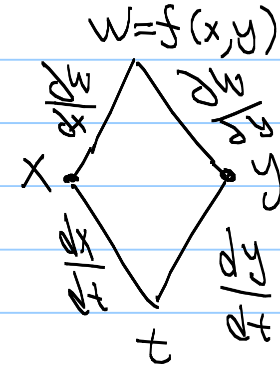
Letting $\Delta t \to 0$ ($\lim_{\Delta t \to 0}$ for both sides) $\Rightarrow$

$$\frac{dw}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + 0 \cdot \frac{dx}{dt} + 0 \cdot \frac{dy}{dt}$$

When $w = f(x,y)$ $\quad x = x(t)$ $\quad y = y(t)$

t is the independent variables

x & y are intermediate variable



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

Ex 1 page 794 $\quad w = xy \quad x = \cos t \quad y = \sin t$

find $\left.\frac{dw}{dt}\right|_{t=\pi/2}$

Note we can rewrite w as $w = \cos t \sin t$ and use product rule to find $\frac{dw}{dt}$

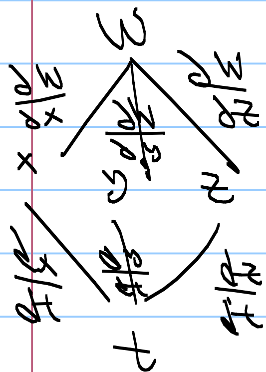
Using Chain Rule

$$\frac{dw}{dt} = w_x \frac{dx}{dt} + w_y \frac{dy}{dt} = y(-\sin t) + x\cos t$$

$$= \sin t(-\sin t) + \cos t \cos t = -\sin^2 t + \cos^2 t$$

$$\left.\frac{dw}{dt}\right|_{\frac{\pi}{2}} = -1^2 + 0^2 = -1$$

Similarly for $W = f(x,y,z)$ $x = x(t)$ $y = y(t)$ $z = z(t)$



Ex 2 page 795

$W = xy + z$ $x = \cos t$ $y = \sin t$ $z = t$

Again we can rewrit W as W(t) without the intermediate variables.

$$\frac{dw}{dt} = W_x \frac{dx}{dt} + W_y \frac{dy}{dt} + W_z \frac{dz}{dt}$$

$$= y(-\sin t) + x(\cos t) + 1(1)$$

$$= -\sin^2 t + \cos^2 t + 1$$

$$\left.\frac{dw}{dt}\right|_{t=0} = 0 + 1^2 + 1 = 2$$

What If $W = f(x,y,z)$ $x = x(r,s)$ $y = y(r,s)$ $z = z(r,s)$

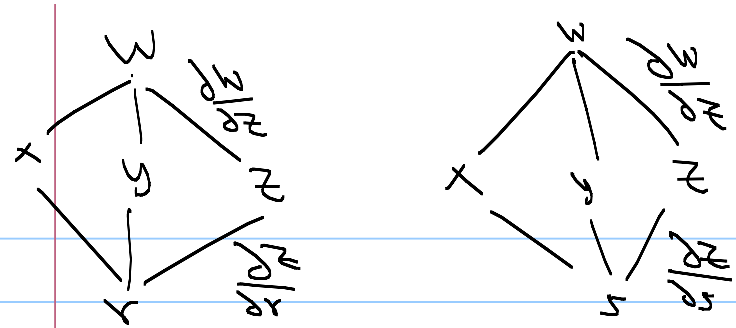
Two independent variables

Here W changes partial (by chang in r, and by change in s)

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

Note it is tempting to cancel $\left(\frac{\partial w}{\cancel{\partial x}} \frac{\cancel{\partial x}}{\partial r}\right)$ partials are not like d !

Similarly for $\frac{\partial w}{\partial s}$



Ex 3, Ex 4 pages 796 + 797

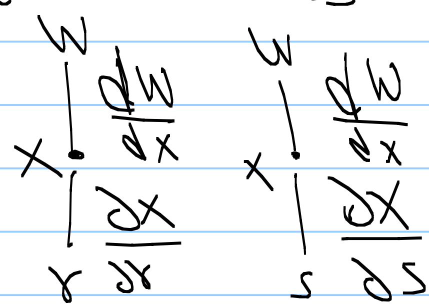
three intermediate

Two intermediate

For one intermediate

$$W = f(x) \qquad x = x(r,s)$$

for $\dfrac{\partial w}{\partial r}$ for $\dfrac{\partial w}{\partial s}$



Note: we can rewrite
W as $w = f(r,s)$

Ex $W = 3rs - \sin(rs)$

$W = 3x - \sin(x)$ where $x = rs$

$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r} = (3 - \cos(x))s = 3s - \cos(rs)s$$

We can use the chain rule for
Implicit differentiation

Ex 5 page 798

Find $\frac{dy}{dx}$ if $y^2 - x^2 = \sin(xy)$

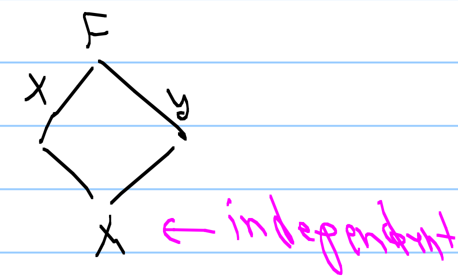
without chain rule $\quad 2y\,y' - 2x = \cos(xy)(y + y'x)$

$$y'(2y - x\cos(xy)) = y\cos(xy) + 2x$$

$$y' = \frac{y\cos(xy) + 2x}{2y - x\cos(xy)}$$

Using chain rule

rewrite as $F(x,y) = 0$ so



$$\frac{dF}{dx} = 0 = F_x \frac{dx}{dx} + F_y \frac{dy}{dx}$$

$$\boxed{\Rightarrow \frac{dy}{dx} = \frac{-F_x}{F_y} \quad F_y \neq 0}$$

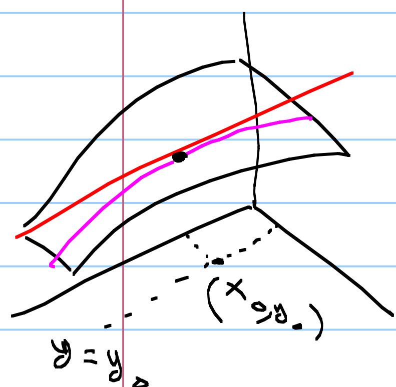
$F(x,y) = y^2 - x^2 - \sin(xy) = 0$

$$\frac{dy}{dx} = \frac{-(-2x - \cos(xy)y)}{2y - \cos(xy)x}$$

Same answer

14.5 Directional Derivatives and Gradiant Vector

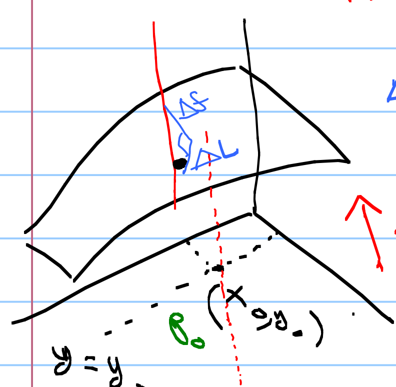
When we learned $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ we fixed
$y = y_0$ & $x = x_0$



$\frac{\partial f}{\partial x}$ was the slope of the curve on the surface traced by the plane $y = y_0$

So it is the rate of change in f in the direction of $\langle 1, 0 \rangle$ (parallel to the x-axis) hence $\boxed{\frac{\partial f}{\partial x}}$

What if we want the rate of change in f in the direction $\langle u_1, u_2 \rangle$



$$\Delta L = \sqrt{(x_0 + su_1 - x_0)^2 + (y_0 + su_2 - y_0)^2} = \sqrt{s^2(u_1^2 + u_2^2)} = s$$

$\uparrow \langle u_1, u_2 \rangle$

$x = x_0 + su_1$
$y = y_0 + su_2$

$$\left(\frac{\partial f}{\partial s}\right)_{u, P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

Ex if $f(x,y) = x^2 + xy$ find the derivative at $P_0(1,2)$ in the direction of

a) $u = j = \langle 0, 1 \rangle$ Note y-direction $\Rightarrow$ x is fixed $\left(\frac{\partial f}{\partial y}\right)$

b) $u = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

a)

$$\left(\frac{\partial f}{\partial s}\right)_{u,P_0} = \lim_{s \to 0} \frac{f(1+s(0),\, 2+s(1)) - f(1,2)}{s}$$

$$= \lim_{s \to 0} \frac{1^2 + 1(2+s) - [1^2 + 1(2)]}{s}$$

$$= \lim_{s \to 0} \frac{3 + s - 3}{s} = 1$$

$$\boxed{\frac{\partial f}{\partial y} = x \Rightarrow \left.\frac{\partial f}{\partial y}\right|_{(1,2)} = 1}$$

b)

$$\left(\frac{\partial f}{\partial s}\right)_{u,P_0} = \lim_{s \to 0} \frac{f(1+s\frac{1}{\sqrt{2}},\, 2+s\frac{1}{\sqrt{2}}) - f(1,2)}{s}$$

$$= \lim_{s \to 0} \frac{\frac{5}{\sqrt{2}}s + s^2}{s} = \frac{5}{\sqrt{2}} \approx 3.5$$

the rate of change of f at $(1,2)$ in the direction of $\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$ is $\frac{5}{\sqrt{2}}$ f grows faster in this direction than in the y direction

How to find directional derivative without limits.

Along the direction $u = \langle u_1, u_2 \rangle$ unit vector

$x = x_0 + s u_1$

$y = y_0 + s u_2$

$f(x, y) = f(s)$

$$\Rightarrow \frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}$$

all along $P_0(x_0, y_0)$ in the direction u

$$= \frac{\partial f}{\partial x} u_1 + \frac{\partial f}{\partial y} u_2$$

$$= \underbrace{\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle}\cdot \langle u_1, u_2 \rangle$$

$$\left(\frac{df}{ds}\right)_{u, P_0} = \nabla f_{P_0} \cdot u$$

∇f is the Gradiant of f

Ex 2 Page 786 $\quad f(x,y) = xe^{y} + \cos(xy)$

Find the derivative at $(2,0)$ in the direction of $v = 3i - 4j$

$\frac{df}{ds} = \nabla f \cdot u \qquad u = \langle \frac{3}{5}, -\frac{4}{5} \rangle$

$\nabla f = \langle e^{y} + -\sin(xy)y,\ xe^{y} - \sin(xy)x \rangle$

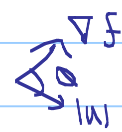
$\nabla f \big|_{(2,0)} = \langle 1-0, 2-0 \rangle = \langle 1, 2 \rangle$

$\frac{df}{ds} = \langle 1, 2 \rangle \cdot \langle \frac{3}{5}, -\frac{4}{5} \rangle = \frac{3}{5} - \frac{8}{5} = \frac{-5}{5} = -1$

Properties of $D_u f = \nabla f \cdot u$ $\qquad$ pag 787

$\nabla f \cdot u = |\nabla f| \cdot |u| \cos\theta$

$= |\nabla f| \cos\theta$



1) ..

2) ..

3) ... $\Longrightarrow \nabla f$ is perpendicular to the levels (curves / surfaces)

(tangents of levels)

Ex 3 page 787 for

$$f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$$

a) direction in which f increases most rapidly
b) " " " " decreases " "
c) " of zero change (perpenindicular to ∇f)

You can always find Normal to a vector $v = \langle v_1, v_2 \rangle$

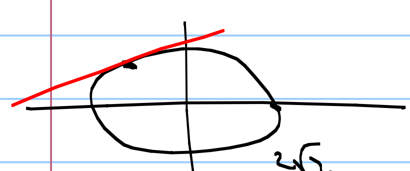
$\langle v_1, v_2 \rangle \cdot \langle u_1, u_2 \rangle = 0 \Rightarrow v_1 u_1 + v_2 u_2 = 0 \Rightarrow u = \langle 0,0 \rangle$ Useless

$\Rightarrow v_1 u_1 = -v_2 u_2$ let $u_1 = 1 \Rightarrow u_2 = -\frac{v_1}{v_2}$

$\therefore$ a unit normal is $\dfrac{\langle 1, -\frac{v_2}{v_1} \rangle}{\sqrt{1 + \frac{v_2^2}{v_1^2}}}$

Ex 4 page 788

eqn of tangent to the ellipse $\frac{x^2}{4} + y^2 = 2$
at $(-2,1)$



Usual way

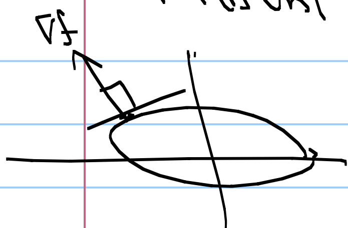
$$y = {+}\cancel{-}\sqrt{2-\frac{x^2}{4}} \qquad y' = \frac{1}{2}\left(2-\frac{x^2}{4}\right)^{-\frac{1}{2}}\left(\frac{-1}{2}x\right)$$

$$y'\Big|_{(-2,1)} = \frac{-1}{4}(-2)\left(2-1\right)^{-\frac{1}{2}} = +\frac{1}{2}$$

tangent Line $\quad y-1 = \frac{1}{2}(x - -2)$

$$y = \frac{1}{2}x + 2$$

Gradient way



$\frac{x^2}{4}+y^2 = 2$ is a level curve

of $f(x,y) = \frac{x^2}{4}+y^2$

$$\nabla f = \left\langle \frac{1}{2}x,\ 2y \right\rangle$$

$$\nabla f\Big|_{(-2,1)} = \langle -1,\ 2\rangle$$

tangent is Normal to ∇f

A Normal to $\nabla f = \langle -1, 2\rangle$ is $\left\langle 1, \frac{-1}{2}\right\rangle = \left\langle 1, \frac{1}{2}\right\rangle$

direction of tangent is $\left\langle 1, \frac{1}{2}\right\rangle$ does not have to be Unit

Point is $(-2,1)$ $\quad\therefore$ tangent line is $x = -2 + 1t$

$$y = 1 + \frac{1}{2}t$$

$$t = x+2 \Rightarrow y = 1 + \frac{1}{2}(x+2)$$

$$y = \frac{1}{2}x + 2$$

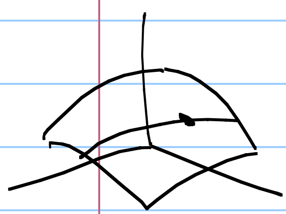
Now Point out Exercise 39 and eqn 6 page 788

Note Rules of ∇f page 789

Ex 5 page 789

Note extension to $f(x, y, z)$ and do Ex 6 page 790

14.6 Tangent planes and Differentials



$z=f(x,y)$ is a Level surface for $f(x,y,z)$ which is $f(x,y,z)=c$

Suppose $r = g(t)\,i + h(t)\,j + K(t)\,K$ is a curve on the surface, then

$$f(\underbrace{g(t), h(t), K(t)}) = c$$

$$\frac{d}{dt}\left(\qquad \right) = \frac{d}{dt}(c)$$

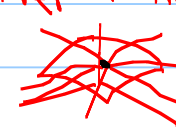
$$\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} + \frac{\partial f}{\partial z}\frac{dK}{dt} = 0$$

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \left\langle \frac{dg}{dt}, \frac{dh}{dt}, \frac{dK}{dt} \right\rangle = 0$$

$$\nabla f \cdot \frac{dr}{dt} = 0 \qquad \frac{dr}{dt} = V_{\text{elocity}}$$

$\therefore\ \nabla f \cdot V = 0 \Rightarrow \nabla f \perp V$ at any point no matter what r is

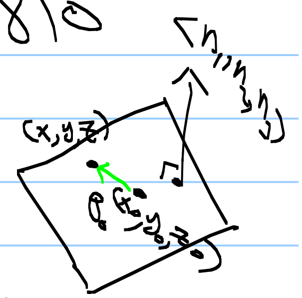
and V is the tangent to the curve and hence the surface



Therefor the lines tangent at P_0 all Lies in the plane with Normal $\nabla f|_{P_0}$ and point P_0

This plane is defined to be the tangent plane

Definition of tangent plane page 810



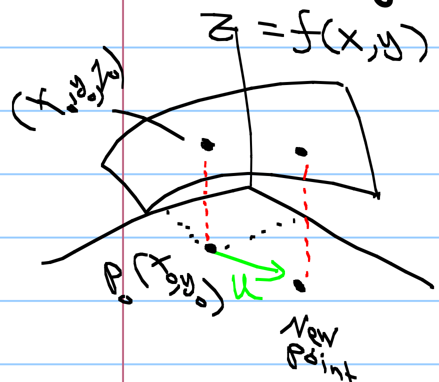
Do Ex 1 page 792

Ex 2 page 793

$\langle (x-x_0),(y-y_0),(z-z_0)\rangle \cdot \langle n_1,n_2,n_3\rangle = 0$

Ex 3 page 793

Estimating Change in a specific direction



If we change the domain by moving a distance $s=ds$ from P_0 in the direction of u, then the exact change in f, $\Delta f = |f(x_0,y_0) - f(x_{new}, y_{new})|$

we can find

However we can estimate the change

$$\frac{df}{ds} = \nabla f\Big|_{P_0} \cdot u \Rightarrow df = \nabla f\Big|_{P_0} \cdot u \, ds \approx \Delta f$$

For single variable

$y=f(x)$ $\qquad \frac{dy}{dx} = f'(x)$

$dy = f'(x)\,dx \approx \Delta y$

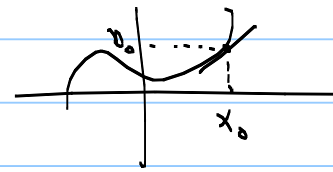
Ex 4 page 812

Linearization of a function of two variables

in single variable

$L(x) = y_0 + f'(x_0)(x - x_0)$

is the tangent line



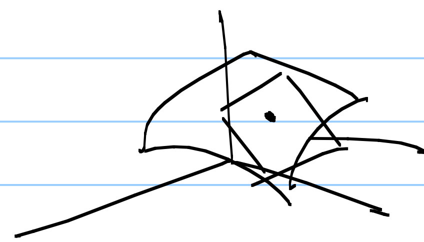
$f(x) \approx L(x)$

$y - y_0 = f'(x_0)(x - x_0)$

$y = y_0 + f'(x_0)(x - x_0)$

$L(x) = y_0 + f'(x_0)(x - x_0)$

in $z = f(x,y)$ the Linearization is the Thangent plane



tangent plane at (x_0, y_0, z_0)

$f(x,y) \approx L(x,y)$

$$z \approx L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

Ex Linearize $z = f(x,y) = x\cos y - ye^x$ at $(0,0,0)$

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

$f(x_0,y_0) = 0$

$f_x = \cos y - ye^x \Rightarrow f_x(0,0) = 1$

$f_y = -x\sin y - e^x \Rightarrow f_y(0,0) = -1$

$\Rightarrow L(x,y) = 0 + 1(x - 0) + -1(y - 0)$

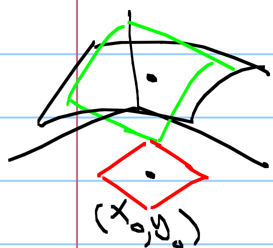
$L(x,y) = x - y$

Note Ex 2 the plane tangent was $x - y - z = 0$

$\Rightarrow z = x - y$ which is Linearization

Ex 5 page 813

The error in standard linear Approximation.



if f has a continuous 1st and 2nd partial derivatives in an open set containing a rectangular region R centered at (x_0, y_0)

Then

$$|E(x,y)| \leq \frac{1}{2} M \left(|x-x_0| + |y-y_0|\right)^2$$

where

$E(x,y)$ is the error of using $L(x,y)$ to Approximate $f(x,y)$

and M is an upper bound for $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ on R

Ex 6 page 796

Total differential of f

We saw (Theorem 3 section 14.3) that if $f(x,y)$ is differentiable then $\Delta f = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta$

$\varepsilon_1, \varepsilon_2 \to 0$ as Δx & $\Delta y \to 0$

$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

if we change x a little bit say from x_0 to $x_0 + dx$ and y from y_0 to $y_0 + dx$ then the change in the linearization is $\Delta L = L(x_0+dx, y_0+dy) - L(x_0,y_0) = f_x(x_0,y_0)\,dx + f_y(x_0,y_0)\,dy$

$\Delta L \approx \Delta f$ $\quad \Delta L$ is called the total differential and it is denoted by $df = f_x\,dx + f_y\,dy$

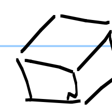
Ex 7 page 815 (do exact change $\pi 1^2(5) - \pi(1.03)^2(4.9) = 0.623$)

Ex 8 page 815

Ex 9 page 816

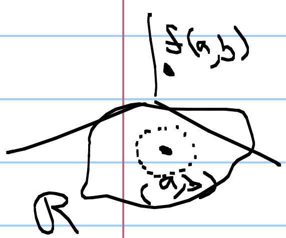
All of the above is extended to f of more than two variables (linearization, error, & differentials)

Ex 10

Note Region for Error is parallel sides 

14.7 Extreme Values and Saddle Points

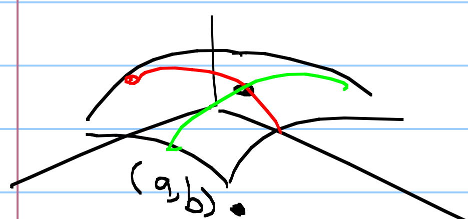
Definitions: if $f(x,y)$ is defined on R containing (a,b) then



1) $f(a,b)$ is a local max if $f(a,b) \geq f(x,y)$ for all points (x,y) in an open disk centered at (a,b)
2) $f(a,b)$ is a local min if $f(a,b) \leq f(x,y)$ for all points (x,y) in an open disk centered at (a,b)
3) $f(a,b)$ is a saddle point if for every open disk centered at (a,b), there are points (x,y) where $f(a,b) > f(x,y)$ and other points (x,y) where $f(a,b) \leq f(x,y)$.

graph $z = x^2+y^2$ $\quad z = -x^2-y^2$ $\quad z = x^2-y^2$

Note Theorem 10



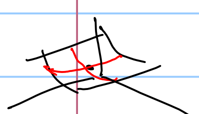
if $f(a,b)$ is Max then $f(x,b)$ has a max at $x=a$

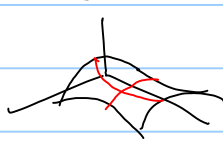
$\Rightarrow f_x(a,b) = 0$

and $f(a,y)$ has a max at $y=b$

$\Rightarrow f_y(a,b) = 0$

Similarly if $f(a,b)$ is a min or a Saddle





Def An interior point (a,b) is a Critical Point if $f_x(a,b)$ and $f_y(a,b)$ are Zero or one or both do not exists

$\therefore$ Theorem 10 $\Rightarrow$ Extram and saddle only occur at Critical Points

Ex 1 $f(x,y) = x^2+y^2-4y+9$ find Local Extrema

$f_x = 2x \quad f_y = 2y-4$ for Critical points $\begin{cases} 2x = 0 \\ 2y-4=0 \end{cases}$

$\Rightarrow x=0 \quad y=2 \quad \therefore$ C.P is $(0,2)$ Max / Min / Saddle ??

$f(0,2) = 5$ → Max / → Min / → Saddle ?

We will lear a test shortly but

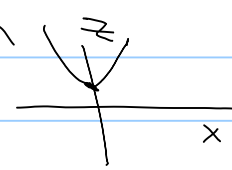
$f(x,y) = x^2 + \underbrace{y^2-4y+4}_{} + \overbrace{4+5}^{9}$ complete the square

$= x^2 + \underbrace{(y-2)^2}_{} + 5$

We Note that both squares has a smallest value of zero $\therefore$ Minimum is 5 in this example it is global (Absolute)

Since we have min at $(0,2)$ then

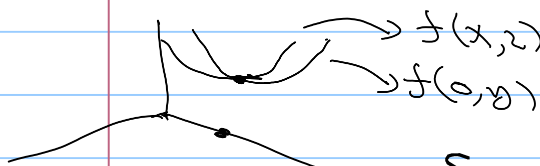
$f(x,2) = x^2+4-8+9 = x^2+5$



has Min at $x=0$, $z=5$

and $f(0,y) = 0+y^2-4y+9$

has Min at $y=2$, $z=5$



is this enough to conclude Min??

No other directions might not have min

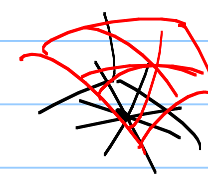
Second derivative test

Theorem 11: Second derivative test for Local extreme values. page 805

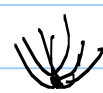
Test: find the discriminant $\begin{vmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{vmatrix}$

$$D = f_{xx} f_{yy} - \underbrace{f_{xy} f_{yx}}_{\text{Same}} \quad \text{at Cr.P}$$

if $D > 0$ Then all curves in all directions at Cr.P



1) curve  downward if $f_{xx} < 0$ and f_{yy} $\Rightarrow$ Max

2) Curve  upward if $f_{xx} > 0$ and f_{yy} $\Rightarrow$ Min

if $D < 0$ then some curves down and some up $\Rightarrow$ Saddle

if $D = 0$ Can't conclude

Why is this So ??

Consider the class of the functions

$$z = ax^2 + bxy + cy^2 = a\left(x^2 + \frac{b}{a}xy\right) + cy^2$$

$$= a\left(x^2 + \frac{b}{a}xy + \left(\frac{1}{2}\frac{b}{a}y\right)^2 - \left(\frac{1}{2}\frac{b}{a}y\right)^2\right) + cy^2$$

$$= a\left(\left(x + \frac{by}{2a}\right)^2 - \left(\frac{1}{2}\frac{b}{a}y\right)^2\right) + cy^2$$

$$= a\left(x + \frac{by}{2a}\right)^2 + -\cancel{a}\frac{b^2y^2}{4a\cancel{^2}} + cy^2$$

$$= a\left(x + \frac{by}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)y^2$$

$$= a\left(x + \frac{by}{2a}\right)^2 + \frac{4ac - b^2}{4a}y^2$$

$z_x = 2ax + by$ $\qquad z_y = 2cy + bx$

$z_{xx} = 2a$ $\qquad z_{yy} = 2c$

$z_{xy} = b$ $\qquad z_{yx} = b$ $\qquad D = 4ac - b^2$

if $4ac - b^2 > 0 \Rightarrow \begin{cases} \text{if } a > 0 \text{ or } c > 0 \Rightarrow \text{Min} \\ \text{if } a < 0 \text{ or } c < 0 \Rightarrow \text{Max} \end{cases}$ same sing in this case

if $4ac - b^2 < 0 \Rightarrow$ one term positive the other is negative $\Rightarrow$ Saddle

if $4ac - b^2 = 0 \Rightarrow$ degenerate term need further examination.

Why is this true in general Taylors App

$$\Delta f \approx \underbrace{f_x}_{\text{Zero at critical}} \Delta x + \underbrace{f_y} \Delta y + \frac{1}{2}f_{xx}(\Delta x)^2 + f_{xy}\Delta x \Delta y + \frac{1}{2}f_{yy}(\Delta y)^2 =$$

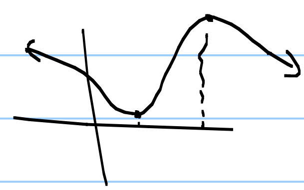
$$\boxed{\begin{array}{l} \text{see } z = x^2 + y^2 \\ z = x^2 - y^2 \\ z = x^2 \end{array}}$$

Ex 3 page 805

Ex 4 page 805

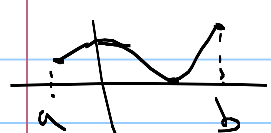
Absoulnte Maxima and Minima on closed Bounded region.

in single Var. for



we don't Know whether the Local extrema are global (absolute). further analysis is needed

but for a function on closed interval [a, b]



There is a min and a max either at C.Ps or at a and b

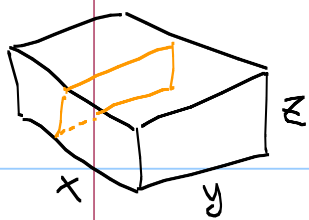
Same with $Z = f(x, y)$

Read 3 steps page 806

Ex 5 page 806

Finding Extrema under constraints

Ex 6 page 807



girth $= 2y + 2z$

Constraint is $x + 2y + 2z = 108$

Maximiz $V = xyz$

Use substitution $z = 54 - y - \frac{1}{2}x$

$\Rightarrow V = f(x,y) = xy\left(54 - y - \frac{1}{2}x\right)$

$= 54xy - xy^2 - \frac{1}{2}x^2y$

$f_x = 54y - y^2 - yx$ $\quad f_{xx} = -y$ $\quad f_{xy} = 54 - 2y - x$

$\parallel$

$f_y = 54x - 2xy - \frac{1}{2}x^2$ $\quad f_{yy} = -2x$ $\quad f_{yx} = 54 - 2y - x$

$\begin{cases} 54y - y^2 - yx = 0 \\ 54x - 2xy - \frac{1}{2}x^2 = 0 \end{cases}$

$\begin{cases} 54 - y - x = 0 \; \cdots (1) \\ 54y - 2xy - \frac{1}{2}x^2 = 0 \; \cdots (2) \end{cases}$ $\Rightarrow$ $y = 54 - x$

$\Rightarrow$ $54x - 2x(54 - x) - \frac{1}{2}x^2 = 0$

$54x - 108x + 2x^2 - \frac{1}{2}x^2 = 0$

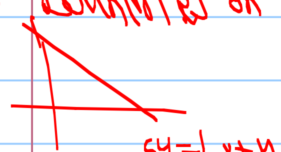
$-54x + \frac{3}{2}x^2 = 0$

$x^2 - 36x = 0$

$x(x - 36) = 0$

$x = 0 \quad x = 36$

Or boundries on x and y



Now as Ex 5

$\Rightarrow y = 54 - 36 = 18$

or $y = 54 - 0 = 54$

$(36, 18)$

$(0, 54)$

$V(36, 18) = 11664$

$V(0, 54) = 0$

Using 2nd Derivative test 11664 is Max $\Rightarrow x = 36, y = 18$

$z = 18$

Solving Extrema problems with constraints using Substitution does not allways comes smooth,

nevertheless what if we can't express one of the variables in terms of the other using constrain

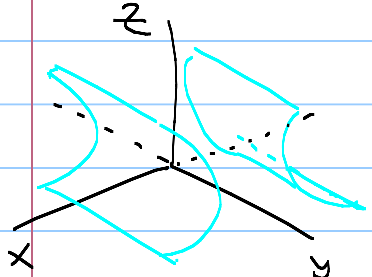
$f(x,y,z)$ constrain $\sin(x) = \ln y \tan^{-1}(z)$

14.8 Lagrange multipliers

14.8 Lagrange Multipliers

It is used to solve extrema problems with constraint since substitution does not always give correct conclusion and sometimes can't solve one variable for the others

Ex 2 Find the closest point on the cylinder $x^2 - z^2 - 1 = 0$ to the origion



$d = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$

$d = f(x,y,z) = \sqrt{x^2+y^2+z^2}$

Minimiz d or d^2 subject to $x^2 - z^2 - 1 = 0$ (constraint)

call d^2 $f(x,y,z) = x^2+y^2+z^2$

Let us try the substitution $z^2 = x^2 - 1$

$\Rightarrow f = x^2+y^2+x^2-1 = 2x^2+y^2-1$

$\frac{\partial f}{\partial x} = 4x \quad \frac{\partial f}{\partial y} = 2y \quad \begin{cases} 4x = 0 \\ 2y = 0 \end{cases} \Rightarrow$ cr. pt is $(0,0)$?

$D > 0$ and $h_{xx} = 4 > 0 \Rightarrow$ Min at $(0,0) \Rightarrow$ Min $h = 0+0-1$

What is wrong with $(0,0)$ it is in the domain of f but not on the cylinder when $|x| \geq 1$

have we used the sub $x^2 = z^2+1$ it would have worked

The Method of Lagrange multiplier Page 815

Ex 2 Objective $f(x,y,z) = x^2+y^2+z^2$

constraint $x^2-z^2-1=0$

$g(x,y,z) = x^2-z^2-1=0$

find x, y, z, and λ for

$\nabla f = \lambda \nabla g$ and $g(x,y,z)=0$

Lagrange multiplier method reduces the problem to solving system of equation

$\nabla f = \langle 2x, 2y, 2z \rangle$

$\nabla g = \langle 2x, 0, -2z \rangle$

$\langle 2x, 2y, 2z \rangle = \lambda \langle 2x, 0, -2z \rangle$

$$\begin{cases} 2x = \lambda 2x & \Rightarrow \lambda = 1 \\ 2y = \lambda(0) & \Rightarrow \boxed{y=0} \\ 2z = -\lambda 2z & \Rightarrow \lambda = -1 \\ x^2-z^2-1=0 \end{cases}$$

$\lambda = 1$	$\lambda = -1$
$2x = 2x \Rightarrow x = x$	$2x = -2x \Rightarrow x = 0$
$2z = -2z \Rightarrow z = 0$	$2z = 2z \Rightarrow z = z$
$x^2 - z^2 - 1 = 0$	$x^2 - z^2 - 1 = 0$
$x^2 - 0^2 - 1 = 0 \Rightarrow x = \pm 1$	$0 - z^2 - 1 = 0$ No solution

$\Rightarrow$ solution of system are $(1,0,0)$ and $(-1,0,0)$

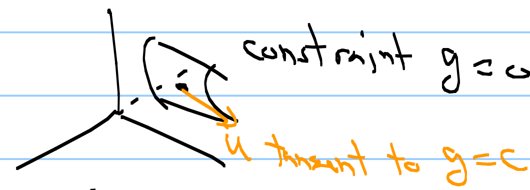
$\therefore$ Min at $(1,0,0)$ and $(-1,0,0)$ it is $f(\pm1,0,0) = 1$

Why this works

the Max/min of $f(x,y,z)$ under $g(x,y,z)=c$
must be at the Level $g=c$
and rate of change of f in any direction along Level $g=c$
must be $\underline{\underline{0}}$

this means: for any direction u tangent to $g=c$

$$\frac{df}{ds}=0$$



$$\Rightarrow \nabla f \cdot u = 0 \Rightarrow \nabla f \perp u$$

So $\nabla f \perp$ to Level g but $\nabla g \perp$ Level g

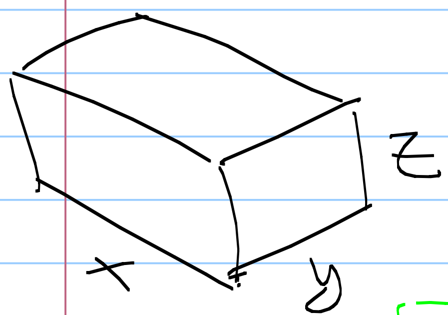
$$\therefore \nabla f \parallel \nabla g \Rightarrow \nabla f = \lambda \nabla g$$

Notes: method does not tell whether a solution is a min or max!

Can't use second derivative test

To determin min or max we need further examination such as comparing values of f at various solutions to the equations

Ex 6 from 14.7 page 807



Maximize $V = xyz$

subject to $x + 2y + 2z = 108$

easier than 27.

Using substitution

$x = 108 - 2y - 2z$

$V = (108 - 2y - 2z)yz$

$V = 108yz - 2y^2 z - 2yz^2$

$f_y = 108z - 4yz - 2z^2 = 0 \qquad (108 - 4y - 2z) = 0 \quad \text{or} \quad z = 0$

$f_z = 108y - 4yz - 2y^2 = 0 \qquad 108 - 4z - 2y = 0 \quad \text{or} \quad y = 0$

$z = 0, y = 0 \qquad (\overset{y}{0}, \overset{z}{0})$

$108 - 4(0) - 2y = 0 \qquad y = 54$

$(54, 0)$

$108 - 4(0) - 2z = 0 \Rightarrow z = 54$

$(0, 54)$

$\begin{cases} 108 - 4y - 2z = 0 \\ 108 - 4z - 2y = 0 \end{cases} \Rightarrow y = 18, \; z = 18$

2nd derivative test

$D > 0 \quad f_{yy} < 0 \Rightarrow \text{Max } V \text{ at } y = 18, \; z = 18 \Rightarrow x = 36$

Using Lagrange method

$f(x,y,z) = xyz$

$g(x,y,z) = x+2y+2z-108 = 0$

$\nabla f = \langle yz, xz, xy \rangle$

$\nabla g = \langle 1, 2, 2 \rangle$

$$\begin{cases} yz = \lambda & \cdots (1) \\ xz = 2\lambda & \cdots (2) \\ xy = 2\lambda & \cdots (3) \\ x+2y+2z-108 = 0 & \cdots (4) \end{cases}$$

Long to solve manualy

Use Maple

eq1 := .... ↵

eq2 :=

⋮

eq4 := ... ↵

solve({eq1, eq2, eq3, eq4}, {x, y, z, l}) ↵

Note Max was at $(36, 18, 18) \Rightarrow \lambda = 18(18) = 324$

$$\nabla f(36,18,18) \overset{??}{=} \overset{\lambda}{324}\, \nabla g(36, 18, 18)$$

$$\langle 324, 648, 648 \rangle = 324 \langle 1, 2, 2 \rangle$$

Ex 3 Page 815

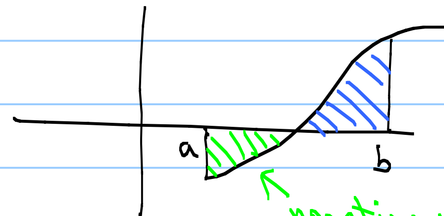
Ex 4 Page 816

15.1 + 15.2 Double Integral

In single var fun $y = f(x)$

$\int_a^b f(x)\,dx$ represents area

between $f(x)$ and $y=0$ (x-axis)



In double var fun $z = f(x,y)$

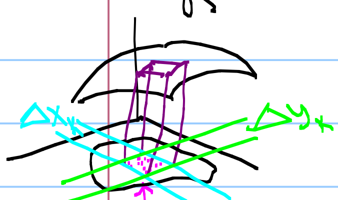
$\iint_R f(x,y)\,dA$ represents volume

between $f(x,y)$ and $z=0$ (xy-plane)

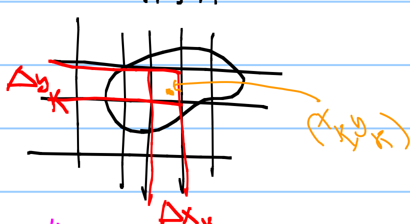
(if part of $f(x,y)$ in $R < 0$ $\Rightarrow$ negative value)

Formal defenition

$$\iint_R f(x,y)\,dA = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k$$



area $= \Delta x_k \Delta y_k = \Delta A_k$



The volume of each column is $\approx f(x_k, y_k)\,\Delta A_k$

is a point in its base

$\therefore$ Volume over R is

$$V \approx \sum_{k=1}^{n} f(x_k,y_k)\,\Delta A_k$$

becomes only one point

As $\|P\| \to 0$ $\Delta x_k \to dx$, $\Delta y_k \to dy$ $\Rightarrow \Delta A_k \to dA$ and $f(x_k, y_k)$

$\therefore$ $V = \iint_R f(x,y)\,dA$ $\quad dA = dy\,dx$ or $dA = dx\,dy$

How to evaluate the double integral?

It is an iterated integral. We integrate twice, once with respect to x and once with respect to y

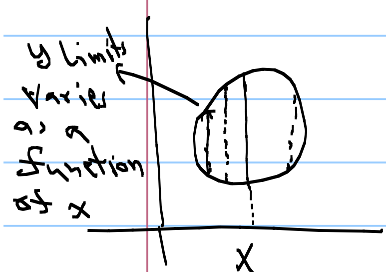
determening the limits of integration.

There are two choices

① fix x and determin the limits of y (if not constant, they will be fun of x)

this gives the inner integral with respect to y (which is the area of the slice).

Then integrate with respect to x from x_{min} to x_{max} (always constant)



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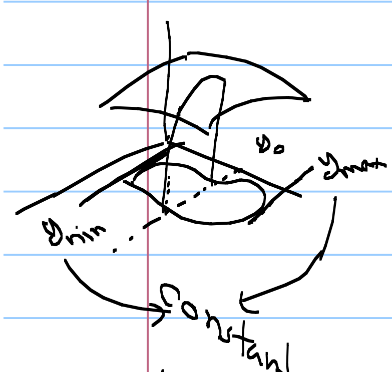
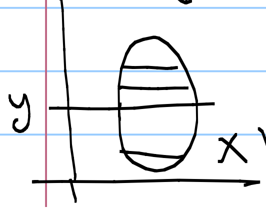
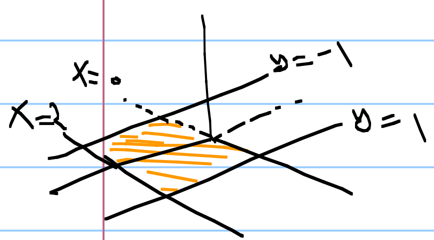
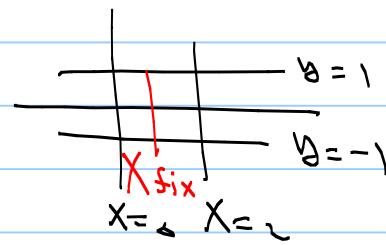




Ex 1 15.1 $\iint_R f(x,y)\,dA$ $f(x,y) = 100 - 6x^2y$

$R: \ 0 \leq x \leq 2 \quad -1 \leq y \leq 1$

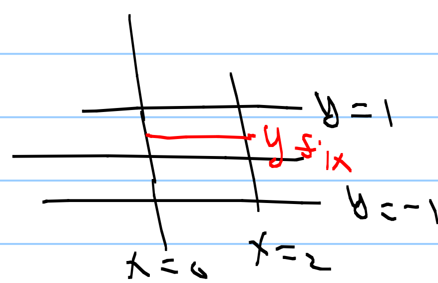




$$\text{fix } x \Rightarrow \int_{x_{min}=\text{const}}^{x_{max}=\text{const}} \int_{y_{min}(x)}^{y_{max}(x)} \quad dy\,dx$$

$$= \int_0^2 \int_{-1}^{1} 100 - 6x^2y \ dy\,dx$$

Or

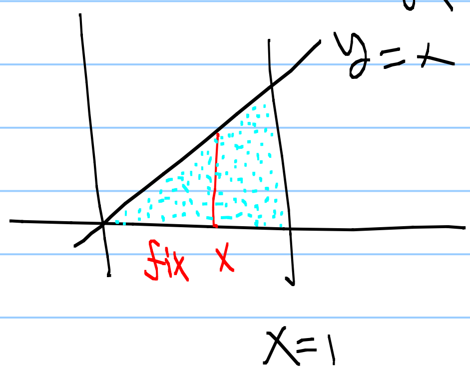


$$\text{fix } y \Rightarrow \int_{y_{min}=\text{const}}^{y_{max}=\text{const}} \int_{x_{min}(y)}^{x_{max}(y)} \quad dx\,dy$$

$$= \int_{-1}^{1} \int_0^2 100 - 6x^2y \ dx\,dy$$

Ex 1 15.2 $Z=f(x,y)=3-x-y$

R bounded by x-axis, $y=x$, $x=1$

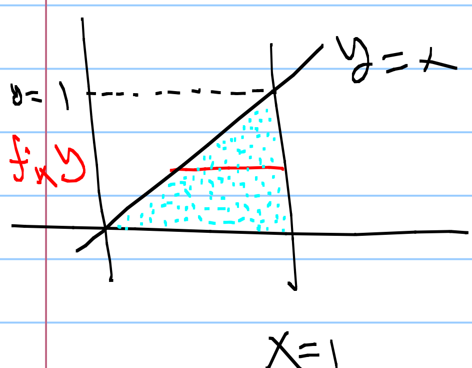


fix X $\displaystyle\int_{x_{min}=\text{Constant}}^{x_{max}=\text{Constant}} \int_{y_{min}(x)}^{y_{max}(x)} dy\,dx$

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=x} 3-x-y \, dy\, dx$$

Or

fix y $\displaystyle\int_{y_{min}=\text{Const}}^{y_{max}=\text{Const}} \int_{x_{min}(y)}^{x_{max}(y)} dx\,dy$

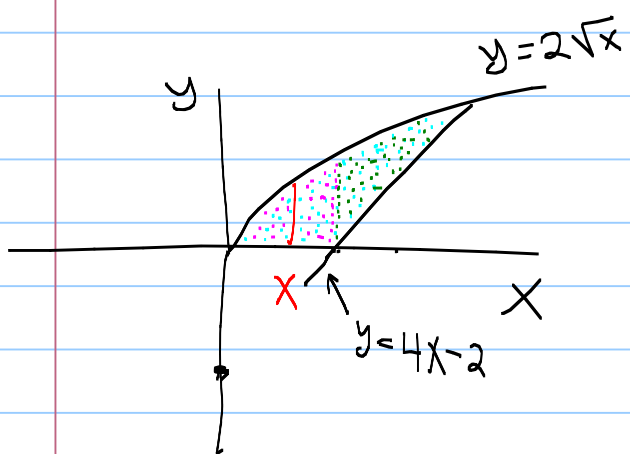


$$= \int_{y=0}^{y=1} \int_{x=y}^{x=1} 3-x-y \, dx\, dy$$

Somtimes fixing one variable leads to two Limits for the other Variable. So you might do two double integrals or try fixing the other Var

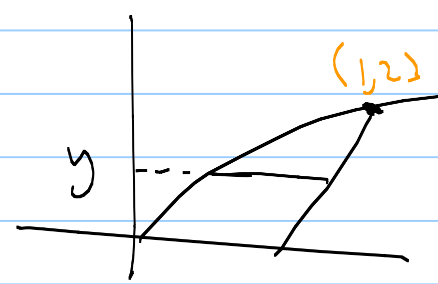
Ex 4 Page 847

EXAMPLE 4 Find the volwne of the wedgelike solid that lies beneath the surface z = 16 - $x2$ - $y2$ and above the region R bounded by the curve Y = 2√x, the line Y = $4x$ - 2, and the x-axis.



Note fixing X gives two Limits of y ⇒ two double integrals

Fixing y will give one Limit of X



$2\sqrt{x} = 4x - 2 \Rightarrow \sqrt{x} = 2x - 1$

$\Rightarrow x = 4x^2 - 4x + 1 \qquad x \geq 0$

$\Rightarrow 4x^2 - 5x + 1 = 0$

$\Rightarrow x = \frac{-b \pm \sqrt{disc}}{2a} = \frac{5 \pm \sqrt{9}}{8} = 1 \text{ or } \cancel{\frac{2}{8}}$

$y = 2\sqrt{1} = 2$

$$\int_{y=0}^{y=2} \int_{x_{min} = y^2/4}^{x_{max} = \frac{y+2}{4}} 16 - x^2 - y^2 \, dx \, dy$$

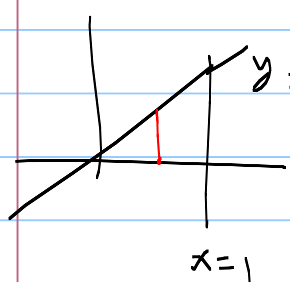
Sometimes integrating in one order is hard or impossible
So we swich order of integration

Ex 2

EXAMPLE 2 Calculate $\iint_R \frac{\sin(x)}{x}\,dA$

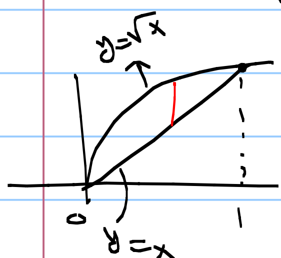
where *R* is the triangle in the xy-plane bounded by the *x-axis,* the line *y* = *x,* and the line

Solving in the order dx dy Not elementary (series)
So try dy dx (fix x)

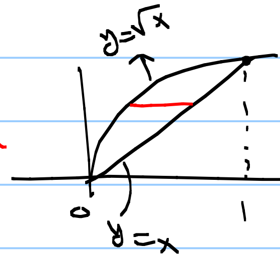


$$\int_{x_{min}=0}^{x_{min}=1}\int_{y_{min}=0}^{y_{max}=x} \frac{\sin(x)}{x}\,dy\,dx$$ easy

Ex Solve $\int_0^1\int_x^{\sqrt{x}} \frac{e^y}{y}\,dy\,dx$ X is fixed between 0 & 1
y between $y=x$ & $y=\sqrt{x}$



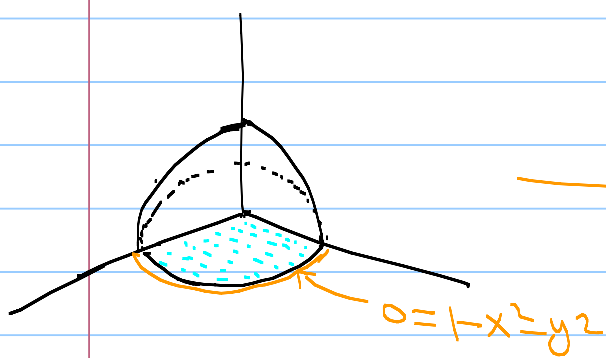
Reverse

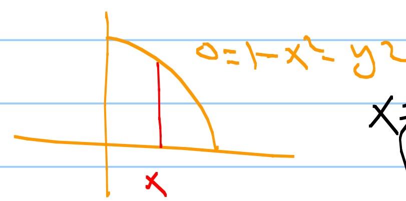


$$\int_{y=0}^{y=1}\int_{x=y^2}^{x=y} \frac{e^y}{y}\,dx\,dy$$

End with this example

Find the volume under $z = 1 - x^2 - y^2$ and above the $z=0$ plane (xy-plane) in the first octant





$$\int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{1-x^2}} 1 - x^2 - y^2 \, dy\, dx$$

$$= \int_{x=0}^{x=1} y - x^2 y - \frac{y^3}{3} \Bigg|_0^{\sqrt{1-x^2}} dx = \int_{x=0}^{x=1} \underbrace{\sqrt{1-x^2} - x^2\sqrt{1-x^2} - \frac{\left(\sqrt{1-x^2}\right)^3}{3} - (0-0-0)\, dx}_{\text{requires trig sub}}$$

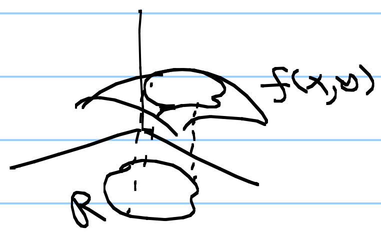
easier with polar coordinates
Section 15.4

15.3 Area by Double Integral

One of the applications of double integrals is to find volume as we saw. Other applications are in Section 15.6 (physics).

In 15.3 we will use double Integral to find <u>area</u> of regions in planes and <u>Average Values</u>

recal



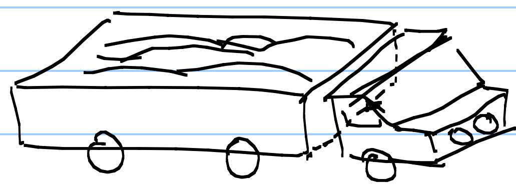
$$V = \iint_R f(x,y)\, dA$$

If $f(x,y) = 1$ then $V = \iint_R 1\, dA$ = are of R

Ex 1, and Ex 2

Average Value: in single Var functions Averag Value $= \dfrac{\int_a^b f(x)\, dx}{b-a}$

In Two Var functions Average Value $= \dfrac{\iint_R f(x,y)\, dA}{\text{area of } R}$



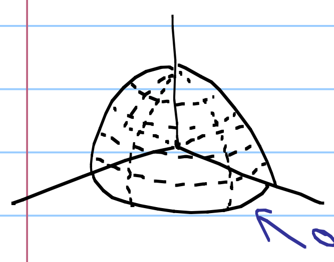
at one instance the water surface function is $f(x,y)$. if the water settle down Its hight is the average

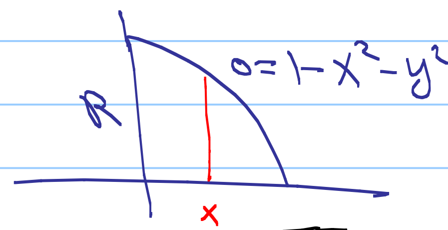
Ex 3

15.4 Double Integrals in Polar Form

Sometimes if we use the polar coordinates the integrals becomes easier.

Ex Find the volume in the first octant under $z = 1 - x^2 - y^2$



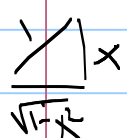


$$\int_0^1 \int_0^{\sqrt{1-x^2}} 1-x^2-y^2 \, dy\, dx$$

$$= \int_0^1 y - x^2 y - \frac{y^3}{3} \Big|_{y=0}^{y=\sqrt{1-x^2}} dx = \int_0^1 \sqrt{1-x^2} - x^2\sqrt{1-x^2} - \frac{\left(\sqrt{1-x^2}\right)^3}{3} \, dx$$

$$= \int_0^1 \sqrt{1-x^2}\,(1-x^2) - \frac{\left(\sqrt{1-x^2}\right)^3}{3} \, dx = \int \frac{2}{3}(1-x^2)^{3/2} \, dx$$

Need this sub. This indicates polar usually easier.



$$\sin\theta = \frac{x}{1} \Rightarrow \frac{2}{3}\int (\cos^2\theta)^{3/2} \cos\theta \, d\theta = \frac{2}{3}\int \cos^4\theta \, d\theta$$

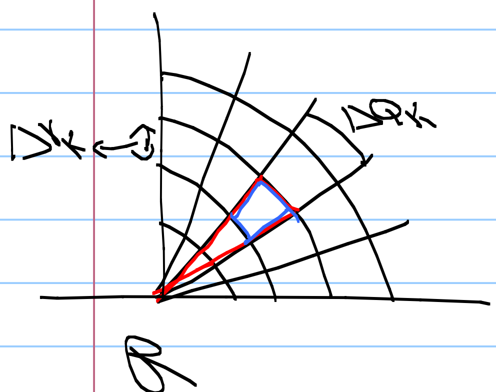
$$= \frac{2}{3}\int \left(\frac{1+\cos 2\theta}{2}\right)^2 = \frac{2}{12}\int 1 + 2\cos 2\theta + \cos^2 2\theta \, d\theta$$

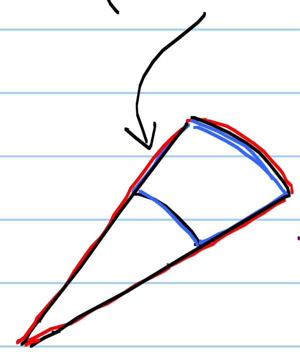
$$= \frac{2}{12}\int 1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2} \, d\theta = \frac{2}{12}\left[\theta + 2\sin(2\theta)\frac{1}{2} + \frac{1}{2}\left(\theta - \sin 4\theta \, \frac{1}{4}\right)\right]$$

$\theta = \sin^{-1}(x) \Rightarrow x = 0 \Rightarrow \theta = 0,\ x = 1 \Rightarrow \theta = \frac{\pi}{2} \Rightarrow \int = \frac{2}{12}\left[\frac{\pi}{2} + 0 + \frac{1}{2}\left(\frac{\pi}{2} - 0\right)\right] - 0 = \pi/8$

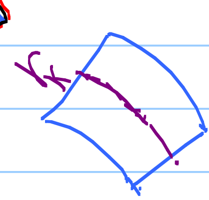
Or use Polar

insted of dividing (partitioning) the region by vertical and horizontal lines (rectangles), divide it by circles and rays (polar rectangles)





area of a polar rectangle $\Delta A_k = ??$



ΔA_k = area of big sector − area of small sector

$$= \frac{\Delta\theta_k (r_k + \frac{1}{2}\Delta r_k)^2}{2} - \frac{\Delta\theta_k (r_k - \frac{1}{2}\Delta r_k)^2}{2}$$

$2\pi \rightarrow \pi r^2$ area

$\theta \rightarrow$?? area

?? area $= \frac{\theta \cancel{\pi} r^2}{2\cancel{\pi}}$

$$= \frac{1}{2}\Delta\theta_k \left[\cancel{r_k^2} + r_k\Delta r_k + \cancel{\frac{1}{4}\Delta r_k^2} - \left(\cancel{r_k^2} - r_k\Delta r_k + \cancel{\frac{1}{4}\Delta r_k^2}\right)\right]$$

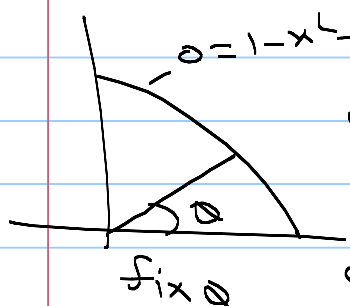
$$\Delta A_k = r_k \Delta r_k \Delta\theta_k \Rightarrow V \approx \sum_{k=1}^{n} \underbrace{f(r_k, \theta_k)}_{\text{hight}} \underbrace{r_k \Delta r_k \Delta\theta_k}_{\text{area of base}}$$

$$V = \iint_R f(r,\theta) \underbrace{r\,dr\,d\theta}_{dA} \quad \text{as } \|P\| \to 0$$

Procedure for finding limits is the same

fix θ

Ex previous $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} 1-x^2-y^2 \, dy \, dx$



$$\int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=1} f(r,\theta) \quad r \, dr \, d\theta$$

$\leftarrow 0=1-x^2-y^2 \Rightarrow x^2+y^2=1 \Rightarrow r=1$

$x = r\cos\theta$
$y = r\sin\theta$
$x^2+y^2 = r$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=1} (1-x^2-y^2)\, r \, dr \, d\theta$$

$$= \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=1} (1-r^2)\, r \, dr \, d\theta = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \frac{r^2}{2} - \frac{r^4}{4} \Big|_{r=0}^{r=1} d\theta$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \frac{1}{2} - \frac{1}{4} - (0) \, d\theta = \frac{1}{4}\theta \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{8}$$

Ex 1 page 855

Area in polar coordinates $\iint_R 1 \, r \, dr \, d\theta$ Ex 2 &

Then Ex 3 & Ex 5 page 856

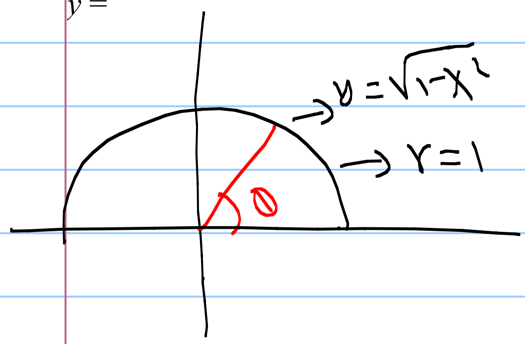
Ex 3

EXAMPLE 3 Evaluate $\iint_R e^{x^2+y^2}\,dy\,dx$

R

where *R* is the semicircular region bounded by the x-axis and the curve $\sqrt{1-x^2}$

y =



$$\to y=\sqrt{1-x^2} \Rightarrow y^2=1-x^2 \Rightarrow x^2+y^2=1 \Rightarrow r^2=1 \Rightarrow r=1$$

$$\to r=1$$

$$\int_{\theta=0}^{\theta=\pi}\underbrace{\int_{r=0}^{r=1} e^{r^2}\, r\,dr}\,d\theta$$

$u=r^2$

$du=2r\,dr$

$dr=\dfrac{du}{2r}$

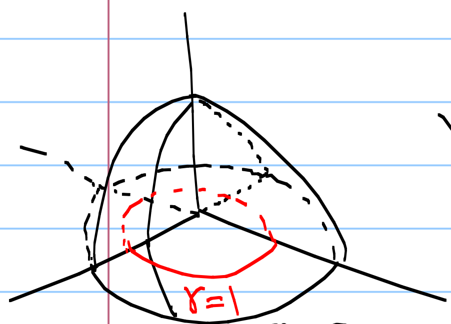
$$\Rightarrow \int_{\theta=0}^{\theta=\pi}\underbrace{\int_{r=0}^{r=1} e^{u}\,\cancel{r}\,\frac{du}{2\cancel{r}}}\,d\theta$$

$$=\int_{\theta=0}^{\theta=\pi}\frac{1}{2}e^{r^2}\Big|_{r=0}^{r=1}\,d\theta=\int_{\theta=0}^{\theta=\pi}\frac{1}{2}(e-1)\,d\theta$$

$$=\frac{1}{2}\left\{e\theta-\theta\Big|_0^{\pi}\right\}=\frac{1}{2}\left\{e\pi-\pi\right\}=\frac{\pi(e-1)}{2}$$

Ex 5

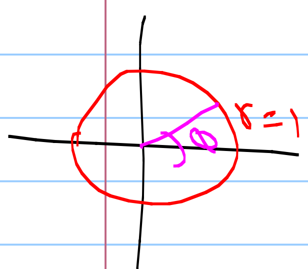
EXAMPLE 5 Find the volume of the solid region bounded above by the paraboloid



$$V=\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} 9-\overset{x^2+y^2}{r^2}\quad r\,dr\,d\theta$$

$0=9-x^2-y^2$

$r=3$



$$=\int_{\theta=0}^{2\pi} \frac{9r^2}{2}-\frac{r^4}{4}\Big|_{r=0}^{r=1}\, d\theta$$

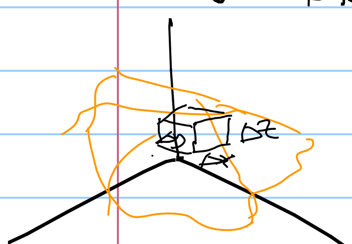
$$=\int_{\theta=0}^{2\pi}\frac{9}{2}-\frac{1}{4}\,d\theta = \frac{17}{4}\theta\Big|_0^{2\pi}$$

$$=\frac{17\pi}{2}$$

15.5 Triple Integrals in Rectangular Coordinates

In Single var we used single integral to find the volume of solids of regular cross sections such as solids of revolution.
In double var we used double integral to find volumes of more general solids.
Trible integrals will allow us to find the volumes of more general shaped solids (and other applications)

If $W = F(x, y, z)$ (Can't graph) then its domain consists of a set in Space D



Partion the set D (solid) into Small cubes then

$$\iiint_D F(x,y,z)\, dV = \lim_{\|\| \to 0} \sum F(x,y,z)\, \Delta V$$

$dV = dz\,dy\,dx$ (in any order) $\quad \Delta V = \Delta z\, \Delta y\, \Delta x$

This triple integral represents several quantities, such as density, depending on what $W = F(x,y,z)$ represents

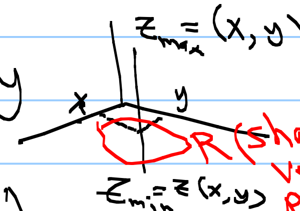
But if $F(x,y,z) = 1$ then the triple integrals are the volume of the solid represented by the set D

Def: The volume of the closed bounded region D in space is $V = \iiint_D 1 \, dV$

$dV = dz \underbrace{dy\, dx}_{dA}$ or any other order

Finding limits of integrations Steps Page 861-862

for the order $dz \underbrace{dy\, dx}_{dA}$ Fix x & y

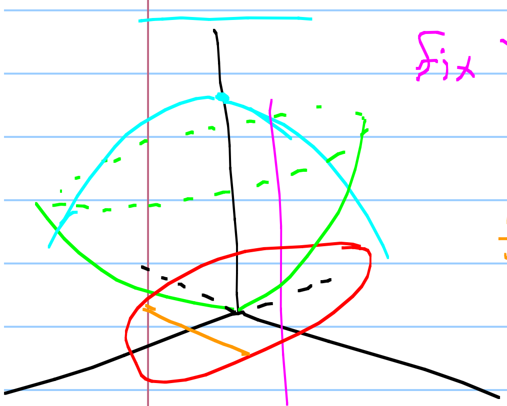


Vertical projection of solid on xy-plane

this gives a line parallel to z so limits of z are $z = f(x,y)$.

Then for $dA = \begin{cases} dy\,dx \\ dx\,dy \end{cases}$ as we learned earlier

EXAMPLE 1 Find the volume of the region *D* enclosed by the surfaces z = *x2* + *3y2* and z = 8 - *x2* - *y2*.



Fix x and y $V = \iint \int_{z_{min} = x^2+3y^2}^{z_{max} = 8 - x^2 - y^2} 1 \; dz \, dA$

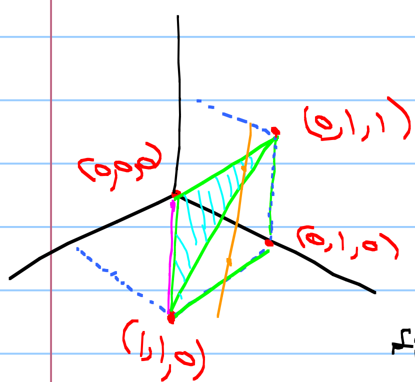
For dA fix x $V = \int_{-2}^{2} \int_{y_{min} = -\sqrt{\frac{8-2x^2}{4}}}^{y_{max} = +\sqrt{\frac{8-2x^2}{4}}} \int_{z_{min} = x^2+3y^2}^{z_{max} = 8-x^2-y^2} dz\, dy\, dx$

Shadow R is $x^2 + 3y^2 = 8 - x^2 - y^2 \Rightarrow 2x^2 + 4y^2 = 8$ ellipse

Ex 3 and Ex 2, Finally Ex 4 for Average Value

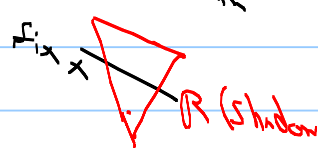
EXAMPLE 2 Set up the limits of integration for evaluating the triple integral of a function $F(x, y, z)$ over the tetrahedron D with vertices (0, 0, 0), (1 , 1,0), (0, 1, 0), and

EXAM PLE 3 Integrate $F(x, y , z) = 1$ over the tetrahedron D in Example 2 in the order



for order $dz\,dy\,dx$

$$\int_{x_{min}=0}^{x_{max}=1}\int_{y_{min}=\text{line}}^{y_{max}=1}\int_{z_{min}=0}^{z_{max}=\text{plane}} F(x,y,z)\,dz\,dy\,dx$$

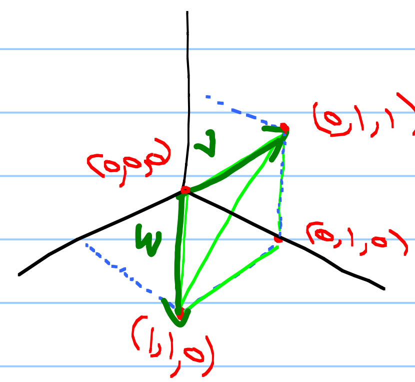


for plane $V=\langle 0,1,1\rangle$ $W=\langle 1,1,0\rangle$

$$V\times W=\begin{vmatrix} i & j & k\\ 0 & 1 & 1\\ 1 & 1 & 0\end{vmatrix}=\langle -1,-1,-1\rangle=\langle -1,1,-1\rangle$$

$$-1(x-0)+1(y-0)+-1(z-0)=0$$

$$\Rightarrow z=y-x$$



for Line $\overset{x\,y}{(0,0)}$ & $\overset{x,y}{(1,1)}$ $\Rightarrow y=x$

So

$$\int_{x_{min}=0}^{x_{max}=1}\int_{y_{min}=x}^{y_{max}=1}\int_{z_{min}=0}^{z_{max}=y-x} F(x,y,z)\,dz\,dy\,dx$$

Volume of tetrahedron is $\int_0^1 \int_x^1 \int_0^{y-x} 1 \, dz\, dy\, dx$

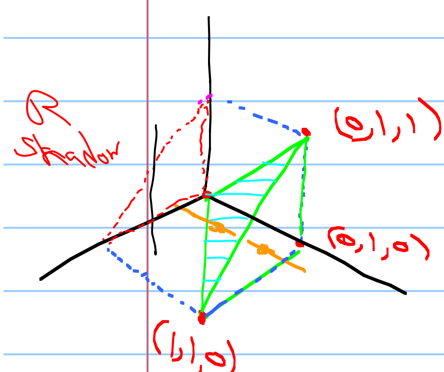
$$= \int_0^1 \int_x^1 z \Big|_0^{y-x} dy\, dx = \int_0^1 \int_x^1 y-x \; dy\, dx$$

$$= \int_0^1 \frac{y^2}{2} - xy \Big|_x^1 dx = \int_0^1 \frac{1}{2} - x - \left(\frac{x^2}{2} - x^2\right) dx$$

$$= \int_0^1 \frac{1}{2} - x + \frac{1}{2}x^2 \, dx = \frac{1}{2}x - \frac{x^2}{2} + \frac{1}{2}\frac{x^3}{3} \Big|_0^1 = \frac{1}{6} \text{ unit}^3$$

for order $dy\, dz\, dx$

fix z and x $\Rightarrow$ Line parallel to y-axis



$$\int_{x_{min}=0}^{x_{max}=1} \int_{z_{min}=0}^{z_{max}=\text{line}} \int_{y_{min}=z+x}^{y_{max}=1} f(x,y,z)\, dy\, dz\, dx$$

For line $(\overset{x}{0}, \overset{z}{1})$ $(\overset{x}{1}, \overset{z}{0})$ slope $\frac{0-1}{1-0} = -1 \Rightarrow z = -1x + b$, $0 = -1(1) + b \Rightarrow b = 1$

$\therefore$ Line $z = -x + 1$

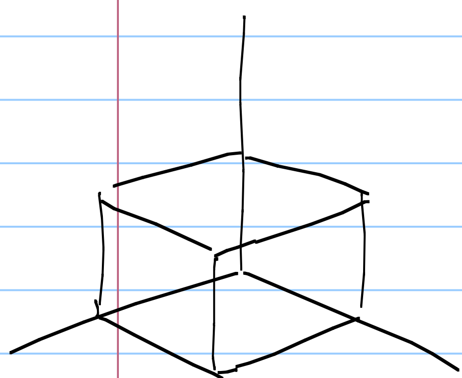
$$\int_{x_{min}=0}^{x_{max}=1} \int_{z_{min}=0}^{z_{max}=-x+1} \int_{y_{min}=z+x}^{y_{max}=1} f(x,y,z)\, dy\, dz\, dx \qquad \iiint 1 \, dy\, dz\, dx = 1/6$$

Average Value of a function in space

Average Value of $F(x,y,z)$ on D $= \dfrac{\iiint_D F(x,y,z)\,dV}{\text{Volume of } D}$

Ex 4 Page 865

EXAMPLE 4 Find the average value of $F(x,y,z) = xyz$ throughout the cubical region
D bounded by the coordinate planes and the planes $x = 2$, $Y = 2$, and $z = 2$ in the



$V = 2(2)(2) = 8$ (cube solid)

Average Value $= \dfrac{\iiint_D xyz\,dz\,dy\,dx}{8}$

$$\int_{x=0}^{x=2}\int_{y=0}^{y=2}\int_{z=0}^{z=2} xyz\,dz\,dy\,dx = 8$$

$\therefore$ Average Value is $\dfrac{8}{8} = 1$

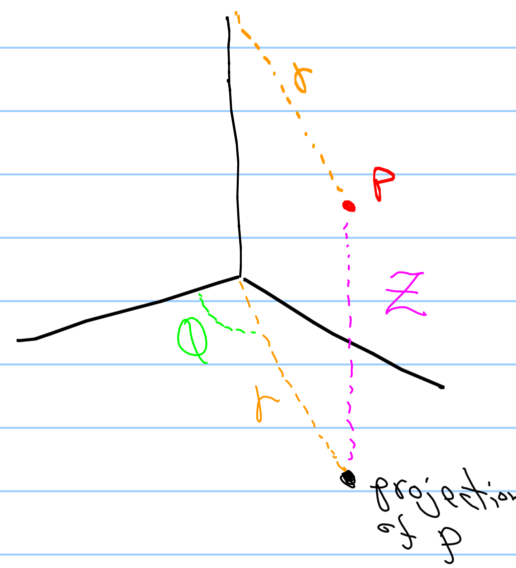
Do Exercise 4 dz dx dy, dy dx dz utilizing Polar to solve the Triple integrals. easier than trig sub.

15.7 Triple Integrals in Cylindrical and Spherical coordinates

Sometimes it is easier to work with problems in these coordinates rather than rectangular. Specially when calculations involve cylinders, cones, or sphers.

Cylindrical coordinates

Def: In cylindrical coordinates, a point P in space is represented by ordered triples (r, θ, z) where r and θ are the polar coordinates of the vertical projection of P on the xy-plane, and z the rectangular vertical coordinate



Rectangular and cylindrical relations

(x, y, z) $\qquad$ (r, θ, z)

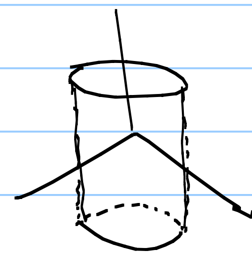
$x = r\cos\theta \qquad r^2 = x^2 + y^2$

$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$

Note: z is the same, and r and θ are what they were in polar

Cylindrical coordinates are good for describing cylinders whose axis is the Z-axis and planes containing the Z-axis.

Ex $r=4$ in cylindrical



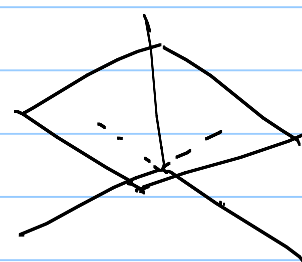
in polar (2D) it was a circle centered at the origion with radius of 4

Ex $\theta=\frac{\pi}{3}$ in cylindrical

in polar (2D) it was a line through the origin

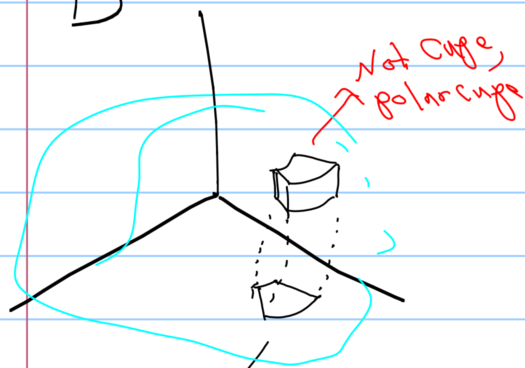
Ex $Z=2$ in cylindrical

is the plane perpendicular to the Z-axis at $Z=2$



Same in rectangular

$$\iiint_D f\, dV = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(r_k, \theta_k, z_k)\, \Delta z_k\, r_k\, \Delta r_k\, \Delta\theta_k$$



Polar rectangle
$\Delta A = r\,\Delta r\,\Delta\theta$

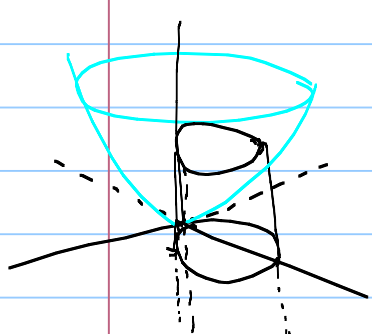
hight · base
$$\Delta V = \Delta z\ \Delta A$$
$$= \Delta z\ r\,\Delta r\,\Delta\theta$$

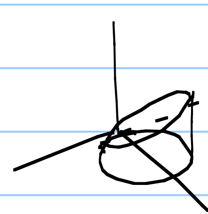
$$\iiint_D f\, dz\, r\, dr\, d\theta$$

$dz\,dr\,d\theta$ is the easiest order for the volum element dV

Ex 1 Page 876

EXAMPLE 1 Find the limits of integration in cylindrical coordinates for
integrating ~~a~~ f
function *I(r, (), z)* over the region *D* bounded below by the plane *z* = 0, laterally by

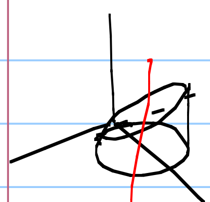




So D is the set bounded bellow by xy-plane $(z=0)$, Laterally by the cylinder $x^2+(y-1)^2=1$, and above by the paraboloid $z=x^2+y^2$

$$\iiint_D F(r,\theta,z)\,dV = \iiint_D F(r,\theta,z)\, r\, dz\, dr\, d\theta$$

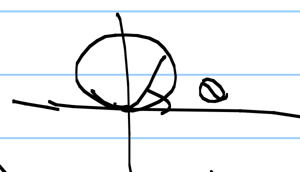
So fix θ and r gives a line parallel to z-axis where it enters the set D is $z_{min} = z(r,\theta)$. Where it exit the set D is $z_{max} = z(r,\theta)$



$$\Rightarrow \int_{z=0}^{z=x^2+y^2} F(r,\theta,z)\, dz = \int_{z=0}^{z=r^2} F\, dz$$

in cylindrical (same as polar)

For $dr\, d\theta$



Fix θ, $r_{min} = f(\theta) = 0$ $\quad r_{max} = f(\theta) = r$ from $x^2+(y-1)^2 = 1$

$$r^2\cos^2\theta + (r\sin\theta - 1)^2 = 1$$

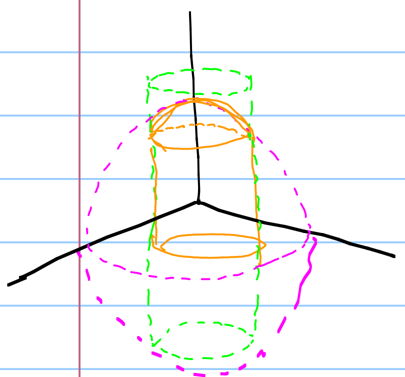
$$r^2\cos^2\theta + r^2\sin^2\theta - 2r\sin\theta + \cancel{1} = \cancel{1}$$

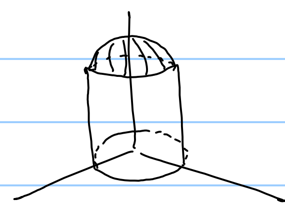
$$r^2 = 2r\sin\theta \Rightarrow r = 2\sin\theta$$

$$\therefore \Rightarrow \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=2\sin\theta}\int_{z=0}^{z=r^2} F(r,\theta,z)\, r\, dz\, dr\, d\theta$$

Exercise 11 page 883

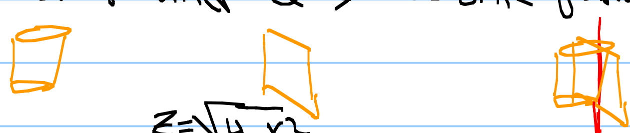
11. Let D be the region bounded below by the plane $z = 0$, above by the sphere $x^2 + y^2 + z^2 = 4$, aod on the sides by the cylinde $x^2 + y^2 = 1$. Set up the triple integrals in cylindrical coordinates that give the volume of D using the following orders of integration.
a. $dz\,dr\,d\theta$ b. $dr\,dz\,d\theta$ c. $d\theta\,dz\,dr$





a) $\iiint 1\, r\,dz\,dr\,d\theta$

1) Fix r and $\theta \Rightarrow$ a Line parallel to z. find $z_{min}(r,\theta)$ and $z_{max}(r,\theta)$



$$\iint \int_{z=0}^{z=\sqrt{4-r^2}}$$

$x^2+y^2+z^2=4$
$z^2=4-r^2$

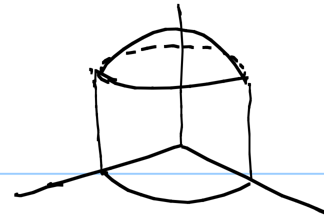
2) for Limits of r and θ, use the projection of D onto xy-plane

fix θ

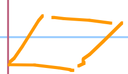
$r_{min}=0$ $r_{max}=1$ $\theta_{min}=0$ $\theta_{max}=2\pi$

$$\therefore \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=0}^{z=\sqrt{4-r^2}} 1\, r\,dz\,dr\,d\theta = \frac{16}{3}\pi - 2\sqrt{3}\,\pi$$

b) $\iiint r\,dr\,dz\,d\theta$



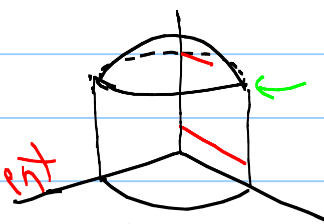
1) Fix z & θ $\Rightarrow$ a line through z-axis parallel to xy-plane



plane perpendicular to z axis

plane containing z-axis

Find $r_{min}(\theta,z)$ and $r_{max}(\theta,z)$



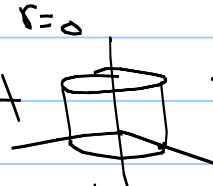
$x^2+y^2+z^2=4$
$x^2+y^2=1$
$z^2=3$
$z=\pm\sqrt{3}$

Note Limits of r are different for two parts of D

$z=-\sqrt{3}$

$$\iint\int_{r=0}^{r=1} r\,dr\,dz\,d\theta + \iint\int_{r=0}^{r=\sqrt{4-z^2}} r\,dr\,dz\,d\theta$$

2) for $dz\,d\theta$ first part



$z_{min}(\theta)=0$ $z_{max}(\theta)=\sqrt{3}$

Second part

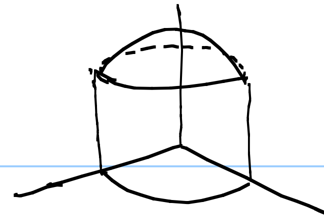
$z_{min}(\theta)=\sqrt{3}$ $z_{max}(\theta)=2$

$\theta_{min}=0$ $\theta_{max}=2\pi$ in both parts

$$\therefore \int_{\theta=0}^{\theta=2\pi}\int_{z=0}^{z=\sqrt{3}}\int_{r=0}^{r=1} r\,dr\,dz\,d\theta + \int_{\theta=0}^{\theta=2\pi}\int_{z=\sqrt{3}}^{z=2}\int_{r=0}^{r=\sqrt{4-z^2}} r\,dr\,dz\,d\theta$$

$$= \sqrt{3}\pi + \left(\frac{16}{3}-3\sqrt{3}\right)\pi = \frac{16}{3}\pi - 2\sqrt{3}\pi$$

c) $\iiint r \, d\theta \, dz \, dr$



1) fix z and r $\Rightarrow$ $\theta_{min}(r,z)=0$ $\quad \theta_{max}(r,z)=2\pi$

2) fix r $\quad z_{min}=0 \quad z_{max}=\sqrt{4-r^2}$

$r_{min}=0 \quad r_{max}=1$

$$\therefore \int_{r=0}^{r=1} \int_{z=0}^{z=\sqrt{4-r^2}} \int_{\theta=0}^{\theta=2\pi} r \, d\theta \, dz \, dz = \frac{16}{3}\pi - 2\sqrt{3}\pi$$

15.8 Substitution in Multiple Integrals.

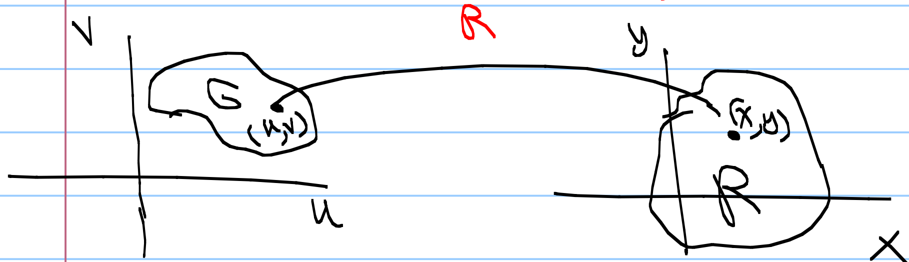
Substitution is used to simplify the integrand, the limits, or both.

If $f(x,y)$, defind on R, is the Image of another region G in the uv-plane by the one-to-one transformation, for interior points, $x = g(u,v)$ and $y = h(u,v)$

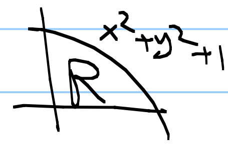
Then $$\iint_R f(x,y)\,dA = \iint_G f(g(u,v), h(u,v))\,|J(u,v)|\,dv$$

where $J(u,v) = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ is a measure of how much the transformation is expanding or contracting the area around a point in G as G is transformed into R

Note: this Implies $\iint_R dy\,dx$ (area of R) $= \iint_G |J|\,du\,dv$



Ex 1 Write the integral $\iint_R f(x,y)\,dx\,dy$

when R is 

Using the transformation $x = r\cos\theta$

$y = r\sin\theta$

Note Polar transformation

without transformation

$$\int_{x=0}^{x=1}\int_{y=0}^{y=\sqrt{1-x^2}} f(x,y)\,dy\,dx$$

since transformation is polar, we should get

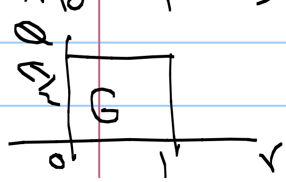
$$\int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r=1} f(r\cos\theta, r\sin\theta)\,\underbrace{r}_{J(r,\theta)}\,dr\,d\theta$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta - r\sin^2\theta = r$$

$$\therefore \iint_R f(x,y)\,dy\,dx = \iint_G f(r\cos\theta, r\sin\theta)\,|r|\,dr\,d\theta$$

For the region G

Boundries of R	transformation	Boundries of G
$x=0 \Longrightarrow$	$r\cos\theta = 0 \Longrightarrow$	$r=0$ or $\cos\theta=0 \Rightarrow \theta=\frac{\pi}{2}$
$y=0 \Longrightarrow$	$r\sin\theta=0 \Longrightarrow$	$r=0$ or $\sin\theta=0 \Rightarrow \theta=0$
$x^2+y^2=1 \Longrightarrow$	$r^2=1 \Longrightarrow$	$r=1$



$$\therefore \int_{x=0}^{x=1}\int_{y=0}^{y=\sqrt{1-x^2}} f(x,y)\,dy\,dx = \int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r=1} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Ex 2 page 888 Ex 3 page 889 Ex 4 page 890

try and hopefully substitution works

Substitution in triple integrals

Same as double integral

Note cylindrical and spherical integrals in 15.7 are special substitution in triple integrals

Ex 5